ALMOST HOLOMORPHIC EMBEDDINGS IN GRASSMANNIANS WITH APPLICATIONS TO SINGULAR SYMPLECTIC SUBMANIFOLDS.

V. Muñoz*, F. Presas** and I. Sols**

** Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, Spain.*

*** Departamento de Álgebra, Universidad Complutense de Madrid, 28040 Madrid, Spain.*

Abstract

In this paper we use Donaldson's approximately holomorphic techniques to build embeddings of a closed symplectic manifold with symplectic form of integer class in the grassmannians $\mathrm{Gr}(r, N)$. We assure that these embeddings are asymptotically holomorphic in a precise sense. We study first the particular case of $\mathbb{CP}^N$ obtaining control on N and by a simple corollary we improve in a sense a classical result about symplectic embeddings [Ti77]. The main reason of our study is the construction of singular determinantal submanifolds as the intersection of the embedding with certain "generalized Schur cycles" defined on a product of grassmannians. It is shown that the symplectic type of these submanifolds is quite more general that the ones obtained by Auroux [Au97] as zeroes of "very ample" vector bundles.

1. Introduction and statement of the main results

Let (M, ω) be a symplectic manifold of integer class, i.e. $[\omega/2\pi] \in H^2(M;\mathbb{R})$ lifts to an integer cohomology class. Such symplectic manifold has an associated line bundle L with first Chern class $c_1(L) = [\omega/2\pi]$, which is equipped with a connection ∇ of curvature $-i\omega$.

In his outbreaking work [Do96] S. Donaldson proved the existence of symplectic submanifolds of M that realize the Poincaré dual of a large enough integer multiple of $[\omega/2\pi]$. These are constructed as zero sets of appropriate sections of $L^{\otimes k}$. This extends a classical result in Kähler geometry saying that L is ample, so $L^{\otimes k}$ has holomorphic sections with smooth holomorphic, and so symplectic, zero sets.

Later on, D. Auroux and R. Paoletti have proved independently an extension of Donaldson's theorem, where now more symplectic submanifolds are constructed as the zero sets of asymptotically holomorphic sections of vector bundles. These bundles are obtained by tensoring an arbitrary complex bundle with large powers of the canonical line bundle L [Au97], [Au99], [Pa99]. In his paper, D. Auroux also shows that, asymptotically, all the sequences of submanifolds constructed from a given vector bundle E are isotopic. (For a summary of these results see for example the review paper [Do98].)

The key idea to understand these works is the concept of ampleness of a complex holomorphic bundle. This concept allows the flexibilization of the bundles in the holomorphic category by means of increasing their curvatures. Donaldson [Do96] has translated the definition of ampleness to the symplectic category. For this he studies the asymptotical behaviour of sequences of sections of the bundles $L^{\otimes k}$. Similarly, the important point in our work is the definition of the concept of asymptotic holomorphicity for sequences of embeddings constructed from very ample linear systems defined over vector bundles more and more twisted.

The change to the non-integrable setting is controlled by this concept. To define it we need to fix a compatible almost complex structure J in (M, ω). So the pair (ω, J) gives a metric g in the tangent bundle. We have a sequence of metrics $g_k = kg$ indexed by integers $k \geq 1$.

Definition 1.1. *Let X be a Hogde manifold with complex structure J_0. Let $\gamma > 0$. A sequence of embeddings $\phi_k : M \to X$ is γ-asymptotically holomorphic if it verifies the following conditions:*

1. *$d\phi_k : T_xM \to T_{\phi_k(x)}X$ has a left inverse θ_k of norm less than γ^{-1} at every point $x \in M$. (The norm is taken with respect to the metric g_k.)*
2. *$|(\phi_k)_*J - J_0|_{g_k} = O(k^{-1/2})$ on the subspace $(\phi_k)_*T_xM$.*
3. *$|\nabla^p \phi_k|_{g_k} = O(1)$ and $|\nabla^{p-1}\bar{\partial}\phi_k|_{g_k} = O(k^{-1/2})$, for all $p \geq 1$.*

A sequence of embeddings is asymptotically holomorphic if there is some $\gamma > 0$ such that it is γ-asymptotically holomorphic.

The first important result is a generalization to the symplectic category of the classical Kodaira's embedding Theorem:

Theorem 1.2. *Given (M, ω) a closed symplectic $2n$-dimensional manifold of integer class endowed with a compatible almost complex structure, there exists an asymptotically holomorphic sequence of embeddings $\phi_k : M \to \mathbb{CP}^{2n+1}$ with $\phi_k^*[\omega_{FS}] = [k\omega]$. Moreover, given two such sequences of embeddings asymptotically holomorphic with respect to two compatible almost complex structures, then they are isotopic for k large enough.*

A sharper, in a sense, result than this has been obtained independently by Bortwick and Uribe in [BU99] using completely different ideas. Their result also obtains control in the symplectic part (equivalently in the metric part) allowing

to obtain asymptotically holomorphic embeddings which are also asymptotically symplectic. Their approach is based on ideas coming from Tian [Ti90] to solve the problem in the Kähler case.

Our main interest for proving Theorem 1.2 is given by the possibility of studying "projective symplectic geometry". We mean by this the study of sequences of asymptotically holomorphic submanifolds, namely obtained as images of asymptotically holomorphic embeddings, in the projective space. The strength of this approach is shown in the following

Theorem 1.3. *Let ϕ_k be an asymptotically holomorphic sequence of embeddings in $\mathbb{CP}^{2n+1}$ with $\phi_k^*[\omega_{FS}] = [k\omega]$ and let $\epsilon > 0$. Let us fix a holomorphic submanifold N in $\mathbb{CP}^{2n+1}$. Then there exists an asymptotically holomorphic sequence of embeddings $\hat{\phi}_k$, at distance at most ϵ in C^r-norm from the initial sequence and verifying that $\hat{\phi}_k(M) \bigcap N$ is symplectic for k large enough.*

With the notations introduced in Section 2 we will precise a little more the precedent result, assuring that $M \bigcap \hat{\phi}_k^{-1}(N)$ is a sequence of asymptotically holomorphic submanifolds.

We will see that this result will imply a projective version of the symplectic Bertini's Theorem proved in [Do96]. But the constructive method could allow to find more general types of symplectic submanifolds. This is shown in a more general situation. For this we generalize Theorem 1.2 to the grassmannian case.

Theorem 1.4. *Let (M, ω) be a closed symplectic $2n$-dimensional manifold of integer class endowed with a compatible almost complex structure. Suppose also that we have a rank r hermitian vector bundle with connection, and that $N > n + r - 1$ and $r(N - r) > 2n$. Then there exist an asymptotically holomorphic sequence of embeddings $\phi_k : M \to \mathrm{Gr}(r, N)$ with $\phi_k^*\mathcal{U} = E \otimes L^{\otimes k}$, where $\mathcal{U} \to \mathrm{Gr}(r, N)$ is the universal rank r bundle over the grassmannian. Moreover, given two such sequences of embeddings asymptotically holomorphic with respect to two compatible almost complex structures, then they are isotopic for k large enough.*

In Section 5 we will take profit of this result to extend the construction of determinantal submanifolds to the symplectic category in the following way.

Definition 1.5. *Let M be a differentiable manifold and let E, F be complex vector bundles over M. Given a morphism of vector bundles $\varphi : E \to F$, the r-determinantal set $\Sigma^r(\varphi)$ is defined as*

$$\Sigma^r(\varphi) = \{x \in M \big| \operatorname{rank} \varphi_x = r\}.$$

In the smooth category we can find for any morphism $\varphi : E \to F$, another morphism $\hat{\varphi} : E \to F$ arbitrarily close to φ in C^p-norm, such that $\Sigma^r(\hat{\varphi})$ is a smooth submanifold in M of codimension $2(r_e - r)(r_f - r)$, where r_e and r_f are the ranks of E and F, respectively (if this number is greater than the dimension of M then the set is empty). There exists a similar result in the algebraic category if the vector bundle $E^* \otimes F$ is very ample. Our objective will be to adapt the algebraic discussion to the symplectic category to prove

Theorem 1.6. *Let (M, ω) be a closed symplectic manifold of integer class. Let E and F be hermitian vector bundles of rank r_e and r_f, respectively. Then, for k large enough, there exists a morphism $\varphi_k : E \otimes (L^*)^{\otimes k} \to F \otimes L^{\otimes k}$ verifying that*

1. $\Sigma^r(\varphi_k)$ *is an open symplectic submanifold of M.*
2. $\operatorname{codim} \Sigma^r(\varphi_k) = 2(r_e - r)(r_f - r)$. *The set of manifolds $\{\Sigma^r(\varphi_k)\}_r$ constitutes a stratified submanifold, called determinantal submanifold.*

Moreover, given two stratified determinantal submanifolds constructed following the process described in the proof then there exists an ambient isotopy making the r-determinantal submanifolds associated to each stratified submanifold isotopic.

Theorem 1.6 was the original motivation of this paper. The idea of studying this kind of submanifolds is inspired in algebraic geometry. Note that in algebraic geometry the manifolds constructed as zeroes of sections of vector bundles have many topological restrictions, namely they satisfy the Lefschetz hyperplane Theorem, their Chern classes are very special, etc. So the set of submanifolds of a given manifold constructed in this way is very special in the set of all the submanifolds. However the determinantal submanifolds are very generic in the set of submanifolds. For instance, every codimension 2 submanifold of an algebraic manifold can be constructed as the determinantal degeneration loci of certain bundle homomorphism [Vo78].

An obvious guess is that in symplectic geometry things are similar. Recall that the most general submanifolds constructed using asymptotically holomorphic techniques, prior to Theorem 1.6 are the Auroux' ones [Au97]. These are zeroes of sections of vector bundles, so its topological properties are very special. In fact, Auroux cannot easily assure that these submanifolds are different from the ones constructed by Donaldson in [Do96]. In Subsection 5.4 we compute some Chern numbers of determinantal submanifolds showing that they are clearly different from the Chern numbers of Auroux' and Donaldson's submanifolds. So the symplectic type, and even the topological type, of the constructed submanifolds is necessarily different. This shows that the class of determinantal submanifolds is far more general.

Remark that, in any case, all the precedents results are obtained by means of twisting vector bundles with large powers of the line bundle L. So the submanifolds constructed in this way are quite special. It would be desirable to avoid this restriction, but this generalization cannot be made with the Donaldson's techniques developped in [Do96].

From a symplectic point of view determinantal submanifolds are also interesting. They constitute a step in the study of singular symplectic submanifolds following the program sketched by Gromov [Gr86]. Donaldson and Auroux have attacked this question in [Do99] and [Au99]. Donaldson studies the local symplecticity of the fibers of asymptotically holomorpic applications $f : \mathbb{C}^n \to \mathbb{C}$ at a neigborhood of a critical point, it is solved by a local perturbation argument. The conclusion of Donaldson's work is that the topological behaviour of that kind of functions is similar to the holomorphic Morse functions. Auroux studies the local symplecticity of asymptotically holomorphic applications $f : \mathbb{C}^2 \to \mathbb{C}^2$ at the neighboorhood of a critical point, showing that are topologically equivalent to one of the two generic models of a holomorphic application [Ar82]. From this point of view Theorem 1.6 can be considered, in part, an extension of these results to generic singularities.

The organization of the paper is as follows. In Section 2 we will give the basic ideas of the Donaldson-Auroux' theory needed in our work and prove Theorem 1.2. In Section 3.2 we prove Theorem 1.3. For this we explain some euclidean notions concerning the estimation of angles between subspaces. In Section 4 we generalize all the discussion to the case of the grassmannian embbedings, proving Theorem 1.4. This allows us to prove Theorem 1.6 in Section 5 and to analyze the topological properties of the constructed submanifolds.

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2. Asymptotically holomorphic embeddings in projective space

As in the introduction, let (M, ω) be a symplectic manifold of integer class with associated line bundle L and a compatible almost complex structure J. In the Kähler setting this line bundle supports a holomorphic structure and it is ample in the algebraic geometry sense, i.e. $L^{\otimes k}$ has a lot of holomorphic sections. This allows to embed M in the projective space $\mathbb{CP}^N$, for some N. In this Section we shall extend this classical result to the symplectic case inspired in the ideas of [Do96], thus proving Theorem 1.2.

2.1. **Asymptotically holomorphic sequences.** In this Subsection we collect the relevant results of the asymptotically holomorphic theory, as stated by D. Auroux in [Au99], that we shall use extensively along this work.

Definition 2.1. *A sequence of sections s_k of hermitian bundles E_k with connections on M is called asymptotically J-holomorphic if there exist constants $(C_p)_{p\in\mathbb{N}}$ such that, for all k, at every point of M, $|s_k| \leq C_0$, $|\nabla^p s_k| \leq C_p$ for all $p \geq 0$, and $|\nabla^{p-1}\bar{\partial} s_k| \leq C_p k^{-1/2}$ for all $p \geq 1$. The norms are evaluated with respect to the metrics g_k.*

In Donaldson's first work on the subject [Do96], $E_k = L^{\otimes k}$. In that work Donaldson imposed an additional condition of improved transversality to the sequence of sections to assure that its zero sets are symplectic submanifolds for k large enough. This condition is stated as follows.

Definition 2.2. *A section s_k of the line bundle $L^{\otimes k}$ is said to be η-transverse to 0 if for every point $x \in M$ such that $|s_k(x)| < \eta$ then $|\nabla s_k(x)| > \eta$.*

If we get an asymptotically J-holomorphic sequence s_k of sections of $L^{\otimes k}$ verifying that all of them are η-transverse to 0, with $\eta > 0$ independent of k then we can assure that $|\partial s_k(x)| > |\bar{\partial} s_k(x)|$ if x is a zero of s_k, for k large enough. A simple linear algebra argument assures that the zeroes of s_k are symplectic submanifolds for k large enough.

In [Au97] D. Auroux extended the notion of transversality to the case of higher rank bundles. Let E be a rank r hermitian bundle with connection.

Definition 2.3. *A section s_k of the bundle $E \otimes L^{\otimes k}$ is η-transverse to 0 if for every $x \in M$ such that $|s_k(x)| < \eta$ then $\nabla s_k(x)$ has a right inverse θ_k such that $|\theta_k| < \eta^{-1}$.*

We name *universal constant* to a number which only depends on the manifold geometry and on the constants involved in the data given to start with, i.e. a number independent of k and the point $x \in M$. Similarly a *universal polynomial* is a polynomial only depending on the geometry of the manifold and on the constants provided in the original data. Donaldson uses highly localized asymptotically holomorphic sections, verifying the following definition.

Definition 2.4. *A sequence of sections s_k of hermitian bundles E_k with connections has Gaussian decay in C^r-norm away from the point $x \in M$ if there exists a universal polynomial P and a universal constant $\lambda > 0$ such that for all $y \in M$, $|s(y)|$, $|\nabla s(y)|_{g_k}$, ..., $|\nabla^r s(y)|_{g_k}$ are bounded by $P(d_k(x, y)) \exp(-\lambda d_k(x, y))$. Here d_k is the distance associated to the metric g_k.*

The starting point for Donaldson's construction is the following existence Lemma.

Lemma 2.5 ([Do96, Au97]). *Given any point $x \in M$, for k large enough, there exist asymptotically holomorphic sections $s^{ref}_{k,x}$ of $L^{\otimes k}$ over M satisfying the following bounds: $|s^{ref}_{k,x}| > c_s$ at every point of a ball of g_k-radius 1 centered at x, for some universal constant $c_s > 0$; the sections $s^{ref}_{k,x}$ have Gaussian decay away from x in C^r-norm.*

Moreover, given a one-parameter family of compatible almost-complex structures $(J_t)_{t\in[0,1]}$, there exist one-parameter families of sections $s^{ref}_{t,k,x}$ which depend continuously on t and satisfy the same precedent properties. □

The proof of this Lemma uses in particular a refined version of Darboux' Theorem taking into account the holomorphic structure, which we also enunciate for later use.

Lemma 2.6 (Lemma 3 in Chapter 3 of [Au99]). *Near any point $x \in M$, for any integer $k \geq 1$, there exist local complex Darboux coordinates $(z_k^1, \dots, z_k^n) = \Phi_k : (M, x) \to (\mathbb{C}^n, 0)$ for the symplectic structure $k\omega$ such that the followings bounds hold universally: $|\Phi_k(y)|^2 = O(d_k(x,y)^2)$ on a ball $B_{g_k}(x, c)$ of universal radius c around x; $|\nabla^r \Phi_k^{-1}|_{g_k} = O(1)$ for all $r \geq 1$ on a ball $B(0, c')$ of universal radius c' around 0; and, with respect to the almost-complex structure J on X and the canonical complex structure J_0 on $\mathbb{C}^n$, $|\bar\partial \Phi_k^{-1}(z)|_{g_k} = O(k^{-1/2}|z|)$ and $|\nabla^r \bar\partial \Phi_k^{-1}|_{g_k} = O(k^{-1/2})$ for all $r \geq 1$ on $B(0, c')$.*

Moreover, given a one-parameter continuous family of compatible $(J_t)_{t\in[0,1]}$ and a continuous family of points $(x_t)_{t\in[0,1]}$, there exists a continuous family of Darboux coordinates $\Phi_{t,k}$ satisfying the same estimates and depending continuously on t.

Proof. In [Au99] the result is stated only for the case $n = 2$ but the proof extends to the case $n > 2$ trivially. □

In [Au99] D. Auroux used three asymptotically holomorphic sections to set up a projection from a symplectic 4-manifold M to $\mathbb{CP}^2$. To control the behaviour of this projection he needs to assure global transversality conditions between the sections. He developes a very useful scheme to pass from local transversality conditions to global ones by means of a globalization process inspired in the results of [Do96]. Now we explain his idea to formalize Donaldson's techniques.

Definition 2.7. *A family of properties $\mathscr{P}(\epsilon, x)_{x\in M, \epsilon>0}$ of sections of bundles over M is local and C^r-open if, given a section s satisfying $\mathscr{P}(\epsilon, x)$, any section σ such that $|\sigma(x) - s(x)|_{C^r} < \eta$ satisfies $\mathscr{P}(\epsilon - C\eta, x)$, where C is universal.*

For example, the property $|s(x)| > \epsilon$ is local and C^0-open. The property that s be ϵ-transverse to 0 at a point x is local and C^1-open.

Proposition 2.8 (Proposition 3 in Chapter 3 of [Au99]). *Let $\mathscr{P}(\epsilon, x)_{x\in M, \epsilon>0}$ be a local and C^r-open family of properties of sections of vector bundles E_k over M. Assume that there exist universal constants c, c', c'' and p such that given any $x \in M$, any small $\delta > 0$, and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections $\tau_{k,x}$ of E_k with the following properties:*

1. *$|\tau_{k,x}|_{C^r, g_k} < c''\delta$.*
2. *The sections $\frac{1}{\delta}\tau_{k,x}$ have Gaussian decay away from x in C^r-norm.*
3. *The sections $s_k + \tau_{k,x}$ satisfy the property $\mathscr{P}(\eta, y)$ for all $y \in B_{g_k}(x, c)$, with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$.*

Then, given any $\alpha > 0$ and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections σ_k of E_k such that $|s_k - \sigma_k|_{C^r, g_k} < \alpha$ and the sections σ_k satisfy $\mathscr{P}(\epsilon, x)$ for all $x \in M$ with $\epsilon > 0$ independent of k.

Moreover, the result holds for one-parameter families of sections, provided the existence of sections $\tau_{t,k,x}$ satisfying properties 1, 2 and 3 and depending continuously on t.

Proof. We only have added the constant c'' to the original statement in Proposition 3 in Chapter 3 of [Au99], which can be absorbed into the formula for η just by enlarging p universally. $\square$

The heart of these techniques is a series of local transversality results which allow to apply Proposition 2.8. These results are based on ideas of complexity of real polynomials coming from the real algebraic geometry. The most powerful result is the following, proved in [Do99].

Definition 2.9. *A function $f : \mathbb{C}^n \to \mathbb{C}^r$ is σ-transverse to 0 at a point $x \in \mathbb{C}^n$ if it verifies at least one the following properties:*

1. $|f(x)| > \sigma$.
2. *$df(x)$ has a right inverse θ such that $|\theta| < \sigma^{-1}$.*

Proposition 2.10. (Theorem 12 in [Do99]) *There exists a universal integer p verifying the following property: for $0 < \delta < \frac{1}{2}$ let $\sigma = \delta(\log(\delta^{-1}))^{-p}$. Let f be a function with values in $\mathbb{C}^r$ defined over the ball $B^+ = B(0, \frac{11}{10}) \subset \mathbb{C}^n$ satisfying the following bounds over B^+,*

$$|f| \leq 1, \ |\bar{\partial} f| \leq \sigma, \ |\nabla \bar{\partial} f| \leq \sigma.$$

Then there exists $w \in \mathbb{C}$ with $|w| < \delta$ such that $f - w$ is σ-tranverse to 0 over the unit ball in $\mathbb{C}^n$. The same result holds for one-parameter families of functions f_t depending continously on $t \in [0,1]$, where we obtain a continuous path $w : [0,1] \to B(0,\delta)$. $\square$

This Proposition is a generalization of Theorem 20 of [Do96], where the case $r = 1$ is proved. Later on D. Auroux in [Au97, Au99] extended the result to the parametric case with $r = 1$ and to the case $r > m$ respectively. Proposition 2.10 covers all the range of possibilities. We mention also that in [IMP99] the result is refined to control the derivatives of the path w_t allowing so a generalization to the contact case of the asymptotically holomorphic techniques.

2.2. **Asymptotically holomorphic embeddings in $\mathbb{CP}^{2n+1}$.** Through this Section we will study the existence of asymptotically holomorphic embeddings of a closed symplectic manifold (M, ω) of integer class and dimension $2n$, endowed with a compatible almost complex structure J, in the projective space $\mathbb{CP}^{2n+1}$. In Section 4 we will develop the techniques to study the more general grassmannian embeddings. We want to prove the following

Theorem 2.11. *Given an asymptotically J-holomorphic sequence of sections s_k of the vector bundles $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ and $\alpha > 0$ then there exists another sequence σ_k verifying that:*

1. $|s_k - \sigma_k|_{C^1, g_k} < \alpha$.
2. *$\mathbb{P}(\sigma_k)$ is an asymptotically holomorphic sequence of embeddings in $\mathbb{CP}^{2n+1}$ for k large enough.*
3. $[k\omega] = [\phi_k^* \omega_{FS}]$.

Moreover, let us have two asymptotically holomorphic sequences ϕ_k^0 and ϕ_k^1 of embeddings in $\mathbb{CP}^{2n+1}$, with respect to two compatible almost complex structures. Then for k large enough, there exists an isotopy of asymptotically holomorphic embeddings ϕ_k^t connecting ϕ_k^0 and ϕ_k^1.

This result gives a proof of Theorem 1.2. We shall proceed by steps to obtain asymptotically holomorphic embeddings of M into $\mathbb{CP}^{2n+1}$.

Definition 2.12. *A sequence of asymptotically J-holomorphic sections s_k of the vector bundles $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ is γ-projectizable if for all $x \in M$, $|s_k(x)| > \gamma$.*

This is a sufficient condition to get a map to $\mathbb{CP}^{2n+1}$ defined as $\phi_k = \mathbb{P}(s_k) : M \to \mathbb{CP}^{2n+1}$, as the γ-projectizability assures that the sections $s_k = (s_k^0, \dots, s_k^{2n+1})$ are not simultaneously zero and so the $\mathbb{P}$ operator is well defined. To get local injectivity we need to impose the following.

Definition 2.13. *Let s_k be a sequence of asymptotically J-holomorphic γ-projectizable sections of the vector bundles $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ for some $\gamma > 0$ and let $0 \le l \le n$. Then s_k is η-generic of order l, with $\eta > 0$, if $|\bigwedge^l \partial \mathbb{P}(s_k)(x)|_{g_k} > \eta$ for all $x \in M$. For $l = 0$ the condition is vacuus.*

We have the following result that will be proved in the following two Subsections.

Proposition 2.14. *Let s_k be an asymptotically J-holomorphic sequence of sections of the vector bundles $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ and $\alpha > 0$. Then there exists another asymptotically holomorphic sequence σ_k verifying:*

1. $|s_k - \sigma_k|_{C^1, g_k} < \alpha$.
2. *σ_k is γ-projectizable and γ-generic of order n for some $\gamma > 0$.*

Moreover, the result holds for one-parameter families of sections where the sections and almost complex structures depend continuously on $t \in [0, 1]$.

With this result we can give the proof of Theorem 2.11.

Proof of Theorem 2.11. We first prove the existence result. The last property is obvious since the hyperplane bundle of $\mathbb{CP}^{2n+1}$ restricts by construction to $L^{\otimes k}$. Let us begin with an asymptotically J-holomorphic sequence σ_k of sections of the bundles $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$. Now we perturb it using Proposition 2.14 to obtain an asymptotically holomorphic sequence s_k with $|s_k - \sigma_k|_{C^1, g_k} < \alpha$, which is γ-projectizable and γ-generic of order n, for some $\gamma > 0$. We have only to check that the sequence $\phi_k = \mathbb{P}(s_k)$ satisfies the required properties in Definition 1.1. More specifically, we shall check that ϕ_k is an immersion of M in $\mathbb{CP}^{2n+1}$, for k large. To get rid of the possible self-intersection we take into account that $2 \dim M < \dim \mathbb{CP}^{2n+1}$ so we can make a generic C^r-perturbation of norm less than $O(k^{-1/2})$ to get an embedding keeping the asymptotic holomorphicity and the genericity of order n.

Choose a point $x \in M$. By a rotation with an element of $U(2n+2)$ acting on $\mathbb{C}^{2n+2}$, we can assure that $s_k(x) = (s_k^0(x), \dots, s_k^{2n+1}(x)) = (s_k^0(x), 0, \dots, 0)$. The transformation is constant on M and only produces a global isometric transformation of $\phi_k(M)$ in $\mathbb{CP}^{2n+1}$. Now using the γ-projectizable property we know that $|s_k^0(x)| \ge \gamma$. By the asymptotically holomorphic bounds of s_k^0 there is a universal c such that $|s_k^0| \ge \gamma/2$ on $B_{g_k}(x, c)$ for all k. We define the application:

$$\begin{aligned} f_k : B_{g_k}(x, c) &\rightarrow \mathbb{C}^{2n+1} \\ y &\mapsto \left(\frac{s_k^1(y)}{s_k^0(y)}, \dots, \frac{s_k^{2n+1}(y)}{s_k^0(y)} \right). \end{aligned}$$

This application can be written as $f_k = \Phi_0 \circ \phi_k$, where Φ_0 is the standard trivialization application in $\mathbb{CP}^{2n+1}$ defined for the chart $U_0 = \{x = [x_0, \dots, x_{2n+1}] \big| x_0 \neq 0\}$. It is well known that Φ_0 is an isometry at the point $[1, 0, \dots, 0]$ if we use the standard metric structure of $\mathbb{C}^{2n+1}$. So we can compute the bounds required in Definition 1.1 using f_k instead of ϕ_k. The asymptotic holomorphicity of s_k and the bound $|s_k^0| \ge \gamma/2$ imply that $|\nabla^p f_k(x)| = O(1)$ and $|\nabla^p \bar{\partial} f_k(x)| = O(k^{-1/2})$, for $p \ge 0$. This proves condition 3 in Definition 1.1.

Now we pass to the issue of the existence of a left inverse. We have the decomposition

$$\bigwedge^n d\phi_k = \bigwedge^n \partial \phi_k + O(k^{-1/2}),$$

where the last term is obtained thanks to $|\bar\partial\phi_k|_{g_k} = O(k^{-1/2})$. By the γ-genericity of order n of ϕ_k, $|\bigwedge^n \partial\phi_k|_{g_k} \geq \gamma$, so $|\bigwedge^n d\phi_k|_{g_k} \geq \gamma/2$ for k large. Let

$$\hat\theta_k = (d\phi_k)^{-1} : (\phi_k)_* T_x M \to T_x M.$$

By the asymptotic holomorphicity condition, we have $|d\phi_k|_{g_k} \leq C_0$ for a universal constant C_0, so $|\hat\theta_k| \leq C\gamma^{-1}$ for another universal constant C. Now define $\theta_k = \hat\theta_k \circ \mathrm{pr}^\perp$, where $\mathrm{pr}^\perp$ is the orthogonal projection of $T_{\phi_k(x)}\mathbb{CP}^{2n+1}$ onto $(\phi_k)_* T_x M$ to get the sought right inverse (reducing γ conveniently).

Finally we compute the norm of $(\phi_k)_* J - J_0 : (\phi_k)_* T_x M \to T_{\phi_k(x)}\mathbb{CP}^{2n+1}$. The expression can be written as

$$(\phi_k)_* J - J_0 = d\phi_k J\hat\theta_k - J_0 = (d\phi_k + J_0 d\phi_k J) J\hat\theta_k = 2\bar\partial\phi_k J\hat\theta_k = O(k^{-1/2}),$$

proving condition 2 in Definition 1.1.

For the isotopy result we follow the ideas of [Au97]. We need the following auxiliary result, which we prove in Subsection 2.5.

Lemma 2.15. *Let $\phi_k : M \to \mathbb{CP}^{2n+1}$ be a sequence of asymptotically holomorphic embeddings with $\phi_k^*[\omega_{FS}] = [k\omega]$. Then there exists a sequence of asymptotically holomorphic sections s_k of $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$, for k large enough, which is γ-projectizable and γ-generic of order n, for some $\gamma > 0$, such that $\phi_k = \mathbb{P}(s_k)$. The same holds for continuous one-parameter families of embeddings and compatible almost complex structures.*

Using Lemma 2.15, we can suppose that $\phi_k^i = \mathbb{P}(s_k^i)$, $i = 0, 1$, where s_k^0 and s_k^1 are two asymptotically holomorphic sequences which are γ-projectizable and γ-generic of order n, $\gamma > 0$. We construct the following family of sequences of asymptotically holomorphic sections:

$$s_k^t = \begin{cases} (1-3t)s_k^0, & \text{with } J_t = J_0, \quad t \in [0, 1/3] \\ 0, & \text{with } J_t = \mathrm{Path}(J_0, J_1), \quad t \in [1/3, 2/3] \\ (3t-2)s_k^1, & \text{with } J_t = J_1, \quad t \in [2/3, 1]. \end{cases}$$

Choose $\alpha > 0$ such that any perturbation of s_k^t of C^1-norm less than α is still $\gamma/2$-projectizable and $\gamma/2$-generic of order n. Applying Proposition 2.14 to s_k^t with this α, we obtain a family σ_k^t which is η-projectizable and η-generic of order n for some $\eta > 0$. We define the family of sequences of asymptotically holomorphic sections:

$$\tau_k^t = \begin{cases} (1-3t)s_k^0 + 3t\sigma_k^0, & t \in [0, 1/3] \\ \sigma_k^{3t-1}, & t \in [1/3, 2/3] \\ (3t-2)s_k^1 + (3-3t)\sigma_k^1, & t \in [2/3, 1]. \end{cases}$$

These are ϵ-projectizable and ϵ-generic of order n sequences of sections, with $\epsilon = \min\{\gamma/2, \eta\}$, so that $\phi_k^t = \mathbb{P}(\tau_k^t)$ are asymptotically holomorphic embeddings (maybe after a further small perturbation to get rid of self-intersections). This implies that ϕ_k^0 and ϕ_k^1 are isotopic for k large enough. □

An important corollary is the existence of symplectic embeddings of M. The following result is similar to [Ti77], but we do not obtain an exact symplectic embedding. On the other hand the dimension of the projective space is controlled in our case.

Corollary 2.16. *Let (M, ω) be a closed symplectic manifold of dimension $2n$ with symplectic form of integer class. Then there exists a symplectic embedding $\phi : M \to \mathbb{CP}^{2n+1}$ verifying that $k\omega = \phi^*\omega_{FS}$, for k large enough.*

Proof. Take a γ-asymptotically holomorphic sequence ϕ_k of embeddings of M in $\mathbb{CP}^{2n+1}$. The key idea is that the linear segment of forms ω_t joining two symplectic forms compatible with a fixed J is symplectic for every t. In our case we have this condition asymptotically. Define the family of 2-forms in M given

by $\omega_t = (1-t)k\omega + t\phi_k^*(\omega_{FS})$, where $t \in [0,1]$. All of them are cohomologous, so to apply Moser's trick [MS94] we only need to prove that they are symplectic. Suppose that there exists t such that ω_t is not symplectic. Then there is a unitary tangent vector $v \in T_xM$, for some $x \in M$, such that $\omega_t(v,w) = 0$, for all $w \in T_xM$. In particular $\omega_t(v, Jv) = 0$. Now expanding this expression we obtain:

$$\begin{aligned}
\omega_t(v,Jv) &= (1-t)k\omega(v,Jv) + t\phi_k^*\omega_{FS}(v,Jv) \\
&= (1-t)kg(v,v) + t\omega_{FS}(d\phi_k v, J_0\partial\phi_k v - J_0\bar{\partial}\phi_k v) \\
&= (1-t)kg(v,v) + tg_{FS}(d\phi_k v, \partial\phi_k v) - tg_{FS}(d\phi_k v, \bar{\partial}\phi_k v) \\
&= (1-t)kg(v,v) + tg_{FS}(d\phi_k v, d\phi_k v) - 2tg_{FS}(d\phi_k v, \bar{\partial}\phi_k v) \\
&= (1-t)kg(v,v) + tg_{FS}(d\phi_k v, d\phi_k v) - tO(k^{-1/2}).
\end{aligned}$$

Thanks to the γ-asymptotically holomorphic embeddings, we have that $g_{FS}(d\phi_k v, d\phi_k v) \geq \gamma^2$. So for k large enough we get a contradiction. □

2.3. **Construction of γ-projectizable sections.** Our objective is to prove the following perturbation result.

Proposition 2.17. *Let s_k be an asymptotically J-holomorphic sequence of sections of vector bundles $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$. Then given $\alpha > 0$, there exists an asymptotically J-holomorphic sequence of sections σ_k verifying:*

1. $|s_k - \sigma_k|_{C^1, g_k} < \alpha$.
2. *σ_k is η-projectizable for some $\eta > 0$.*

Moreover, the result can be extended to continuous one-parameter families of asymptotically J_t-holomorphic sequences $s_{t,k}$ obtaining continuous one-parameter families of sections $\sigma_{t,k}$ verifying the two precedent conditions.

Proof. The result is a simple generalization of Proposition 1 in [Au99] where the result for 4-manifolds is proved. The high dimensional case can be treated with the same techniques.

We will proceed by using the globalization argument described in Proposition 2.8. First we deal with the non-parametric case. For this we define the local and C^0-open property $\mathscr{P}(\epsilon, x)$ as $|s_k(x)| > \epsilon$. Let $\delta > 0$. We only need to find for a point $x \in M$ a section $\tau_{k,x}$ with Gaussian decay away from x, assuring that $s_k + \tau_{k,x}$ verifies $\mathscr{P}(\eta, y)$ in a ball of universal g_k-radius c, with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$, c' and p universal constants.

For this choose a section $s_{k,x}^{\mathrm{ref}}$ verifying the conditions of Lemma 2.5. Then we select $c = 1$ (obviously, universal). The lower bound of $s_{k,x}^{\mathrm{ref}}$ in the ball $B_x = B_{g_k}(x, 1)$ let us define the application

$$f_{k,x} = \frac{s_k}{s_{k,x}^{\mathrm{ref}}} : B_x \to \mathbb{C}^{2n+2}.$$

Using the lower bound of $s_{k,x}^{\mathrm{ref}}$ together with the asymptotic holomorphicity of s_k is easy to show that

$$|f_{k,x}| < C,\ |\bar{\partial} f_{k,x}| < Ck^{-1/2},\ |\nabla\bar{\partial} f_{k,x}| < Ck^{-1/2}, \tag{1}$$

where C is a universal constant. With the aid of Lemma 2.6 we can build $f_k = f_{k,x} \circ \Phi_k^{-1}$ defined on a fixed ball $B(0, c') \subset \mathbb{C}^n$. Scaling the coordinates by a universal constant $\frac{11}{10}(c')^{-1}$ we can suppose that f_k is defined on B^+. In this ball, the bounds (1) yield

$$|f_k| < C_0,\ |\bar{\partial} f_k| < C_0 k^{-1/2},\ |\nabla\bar{\partial} f_k| < C_0 k^{-1/2}, \tag{2}$$

where C_0 is a universal constant. The application $g_k = \frac{1}{C_0} f_k$ is in the hypothesis of Proposition 2.10 and then there exists, for k large enough, a number $w_k \in B(0,\delta)$ such that $|g_k - w_k| > \sigma = \delta(\log(\delta^{-1}))^{-p}$. Therefore $|f_k - C_0 w_k| > C_0\sigma$ on B. Now

define $\tau_{k,x} = -C_0 w_k \otimes s^{\mathrm{ref}}_{k,x}$, so that $|\tau_{k,x}|_{C^r,g_k} < c''\delta$, for some universal constant c''. Using the lower bound of $s^{\mathrm{ref}}_{k,x}$ we obtain that $|s_k + \tau_{k,x}| \geq c'\delta(\log(\delta^{-1}))^{-p}$, with c' and p universal constants. Then Proposition 2.8 applies and the proof is concluded in the non-parametric case.

The globalization to the one-parameter case is trivial because all the ingredients in the proof can be easily chosen in a continuous way. □

2.4. **Inductive construction of sections γ-generic of order l.** Now we study the problem of perturbing a γ-projectizable sequence of sections to achieve genericity of order n. We shall do this in steps. The result to be proved is the following

Proposition 2.18. *Let s_k be an asymptotically J-holomorphic sequence of sections of the vector bundles $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ which is γ-projectizable and γ-generic of order l. Then given $\alpha > 0$, there exists an asymptotically J-holomorphic sequence of sections σ_k verifying:*

1. $|s_k - \sigma_k|_{C^1,g_k} < \alpha$.
2. *σ_k is η-generic of order $l+1$ for some $\eta > 0$.*

Moreover, this can be extended to continuous one-parameter families of asymptotically J_t-holomorphic sequences $s_{t,k}$ obtaining continuous one-parameter families of sections $\sigma_{t,k}$ verifying conditions 1 and 2.

Proof. We construct local 1-forms to control the perturbations. For this at a neighborhood of a point $x \in M$ we fix local complex Darboux coordinates $(z^1_k, \dots, z^n_k)$ using Lemma 2.6. As in proof of Theorem 2.11, by applying a unitary transformation to $\mathbb{C}^{2n+2}$, we can suppose that $s_k(x) = (s^0_k(x), 0, \dots, 0)$. Also there exists a ball with center x and universal g_k-radius c on which $|s^0_k| \geq \gamma/2$. We define, following Auroux' notations [Au99], a local basis of asymptotically holomorphic 1-forms:

$$\mu^j_k = \partial\left(\frac{z^j_k s^{\mathrm{ref}}_{k,x}}{s^0_k}\right),$$

where $s^{\mathrm{ref}}_{k,x}$ are given by Lemma 2.5. They have Gaussian decay away from x thanks to the behaviour of $s^{\mathrm{ref}}_{k,x}$. At x they form an orthonormal basis of $T^*_x M$. We use the trivialization Φ_0 to define the application

$$\begin{array}{rccl} (3) \qquad & f_k : B_{g_k}(x,c) & \to & \mathbb{C}^{2n+1} \\ & y & \mapsto & \left(\dfrac{s^1_k(y)}{s^0_k(y)}, \dots, \dfrac{s^{2n+1}_k(y)}{s^0_k(y)}\right), \end{array}$$

which is almost an isometry on $B_{g_k}(x,c)$.

The case $l = 0$ without parameters is the easiest. We say that a section $\gamma/2$-projectizable verifies $\mathscr{P}(\epsilon, x)$ if $|\partial\phi_k(x)| > \epsilon$. This property is local and open in C^1-sense. We are going to apply Proposition 2.8 to assure the existence of a η-generic of order 1 sequence of sections arbitrarily near the given s_k in C^1-norm, for some $\eta > 0$. For this let $0 < \delta < \gamma/2c''$, c'' a universal constant whose precise value will appear later. We have to build a local perturbation $\tau_{k,x}$ with $|\tau_{k,x}| < c''\delta$ and Gaussian decay to achieve the property $\mathscr{P}(\eta, y)$ in a neighborhood of x of universal g_k radius c, with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$. (As we only perturb with sections of C^0-norm less than $\gamma/2$ we can assure that all the sections still have the property $\gamma/2$-projectizable.)

Fixing $x \in M$, we have the applications f_k of (3). It is easy to check that there is a ball of universal radius c_0 where

$$\partial f_k = (u^{11}_k \mu^1_k + u^{12}_k \mu^2_k + \cdots + u^{1n}_k \mu^n_k, \dots, u^{2n+1,1}_k \mu^1_k + \cdots + u^{2n+1,n}_k \mu^n_k),$$

for some u_k^{ij}. Then we obtain an application $u_k : B_{g_k}(x, c_0) \to \mathbb{C}^{n\times(2n+1)}$. Using a complex Darboux chart we can trivialize $B_{g_k}(x, c_0)$ to obtain (scaling the coordinates by an appropriate universal constant C) an application $\hat{u}_k : B^+ \to \mathbb{C}^{n\times(2n+1)}$ which is asymptotically holomorphic by construction. So we can apply Proposition 2.10 to get $w'_k \in \mathbb{C}^{n\times(2n+1)}$ such that $|\hat{u}_k - w'_k| > \eta = \delta(\log(\delta^{-1}))^{-p}$ on B, where $|w'_k| < \delta$. Rescaling and passing to the manifold we obtain that $|u_k - Cw'_k| > C\delta(\log(\delta^{-1}))^{-p}$. We denote $w_k = Cw'_k$ and define the section $\tau_{k,x} = -(0, w_k^{11} z_k^1 s_{k,x}^{\mathrm{ref}} + w_k^{12} z_k^2 s_{k,x}^{\mathrm{ref}} + \cdots + w_k^{1n} z_k^n s_{k,x}^{\mathrm{ref}}, \dots, w_k^{2n+1,1} z_k^1 s_{k,x}^{\mathrm{ref}} + \cdots + w_k^{2n+1,n} z_k^n s_{k,x}^{\mathrm{ref}})$ of $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$. This section verifies the properties required in Proposition 2.8.

To check the one-parameter case we have only to get a continuous family of unitary transformations verifying that $s_{t,k}(x) = (s_{t,k}^0(x), 0, \dots, 0)$ for all $t \in [0,1]$. This is clearly possible because of the contractibility of $[0,1]$.

Now we pass to the case $l > 0$. We define the following property for sections s_k which are $\gamma/2$-projectizable and $\gamma/2$-generic of order l. A section s_k has the property $\mathscr{P}(\epsilon, x)$ if $|\bigwedge^{l+1} \partial\mathbb{P}s_k(x)| > \epsilon$. This property is local and open in C^1-sense. For applying Proposition 2.8 we need to build, for $0 < \delta < \gamma/2c''C$, a local perturbation $\tau_{k,x}$ with $|\tau_{k,x}| < c''\delta$ and Gaussian decay with the property $\mathscr{P}(\eta, y)$ in a neighborhood of x of universal g_k radius c, with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$. (Here C is the constant of the C^1-openness of $\mathscr{P}(\epsilon, x)$ in Definition 2.7.) We define f_k as in (3). Then it is easy to see that there exists a universal constant c such that

$$\frac{|\bigwedge^{l+1} \partial\mathbb{P}(s_k)|}{|\bigwedge^{l+1} \partial f_k|} > 1/2$$

on $B_{g_k}(x, c)$. So we can do the computations for the applications f_k. By a unitary transformation in $U(2n+1)$ (on $\mathbb{C}^{2n+2}$ fixing $(1, 0, \dots, 0)$) and other in $U(n)$ (on the complex Darboux coordinate chart) we can assure that

$$\partial f_k(x) = \begin{pmatrix} u_k^{11}(x) & 0 & \dots & & & \dots & 0 \\ 0 & u_k^{22}(x) & 0 & \dots & & & 0 \\ & & & & & & \\ 0 & \dots & \ddots & 0 & \dots & & 0 \\ 0 & \dots & 0 & u_k^{nn}(x) & 0 & \dots & 0 \end{pmatrix}, \tag{4}$$

where $|u_k^{11}(x) \cdots u_k^{ll}(x)| > \gamma/C'$, C' a universal constant. Shrinking c if necessary we can assure that $|(\partial f_k^1 \wedge \cdots \wedge \partial f_k^l)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l}| > \gamma/2C'$ for all the points of the ball $B_{g_k}(x, c)$, where we denote by $(\partial f_k^1 \wedge \cdots \wedge \partial f_k^l)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l}$ the component of $\partial f_k^1 \wedge \cdots \wedge \partial f_k^l$ in the direction of $\mu_k^1 \wedge \cdots \wedge \mu_k^l$. This l-form is an element of the basis composed by the l-wedge products of the 1-forms $\mu_k^1, \dots, \mu_k^n$. In matrix form we are denoting the order l left upper minor of ∂f_k. Now we construct the $(l+1)$-form

$$\theta_k(y) = (\partial f_k^1 \wedge \cdots \wedge \partial f_k^l)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l} \wedge \mu_k^{l+1}.$$

We can suppose that $|\theta_k| > c_s\gamma$ with $c_s > 0$ a universal constant. We also consider the following family of $(l+1)$-forms

$$M_k^p = (\partial f_k^1 \wedge \cdots \wedge \partial f_k^l \wedge \partial f_k^p)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l \wedge \mu_k^{l+1}}, \quad l+1 \le p \le 2n+1.$$

These forms are components of $\bigwedge^{l+1} \partial f_k$. If we perturb so that the norm of $M_k = (M_k^{l+1}, \dots, M_k^{2n+1})$ is bigger than $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ then we have finished because if $|M_k| > \eta$ then $|\bigwedge^{l+1} \partial f_k| > C_0\eta$ where C_0 is again a universal constant (using that the basis $\{\mu_k^{i_1} \wedge \cdots \wedge \mu_k^{i_{l+1}}\}_{1 \le i_1 < \cdots < i_{l+1} \le n}$ is almost orthogonal on the ball $B_{g_k}(x, c)$, in fact orthogonal at x).

We define the following sequence of asymptotically holomorphic applications,

$$g_k = (g_k^{l+1}, \dots, g_k^{2n+1}) = \left(\frac{M_k^{l+1}}{\theta_k}, \dots, \frac{M_k^{2n+1}}{\theta_k} \right).$$

So we obtain, scaling the coordinates by universal constants if necessary, $\hat{g}_k : B^+ \to \mathbb{C}^{2n+1-l}$ which is asymptotically holomorphic thanks to the lower bound of θ_k and to the asymptotic holomorphicity of M_k and θ_k. We have that $n < 2n+1-l$ and so we can find $|w_k| < \delta$ such that $|g_k - w_k| > \eta = \delta(\log(\delta^{-1}))^{-p}$. Thus we obtain that $|(M_k^{l+1} - w_k^{l+1}\theta_k, \dots, M_k^{2n+1} - w_k^{2n+1}\theta_k)| > c_s\gamma\eta$. Recall that all the constants depend on γ and the asymptotic holomorphicity constants of s_k, so they are independent of x and k. The perturbation $-(w_k^{l+1}\theta_k, \dots, w_k^{2n+1}\theta_k)$ is achieved by adding the section $\tau_{k,x} = -(0, \overset{(l)}{\dots}, 0, w_k^{l+1} z_k^{l+1} s_{k,x}^{\mathrm{ref}}, \dots, w_k^{2n+1} z_k^{l+1} s_{k,x}^{\mathrm{ref}})$ to s_k. This section verifies the Gaussian decay bounds required in Proposition 2.8 and $|\tau_{k,x}|_{C^1,g_k} < c''\delta$ for some universal constant c''. This completes the proof in the non-parametric case.

Now we pass to the one-parameter case. With appropriate continuous unitary transformations, we may assume that $s_{t,k}(x) = (s_{t,k}^0(x), 0, \dots, 0)$ and that $\partial f_{t,k}(x)$ is written as in (4). The interval $[0,1]$ may be split in a finite number of subintervals $[t_i, t_{i+1}]$ such that, for every $x \in M$ and for each of the subintervals, there is a fixed order l minor of $\partial f_{t,k}(x)$ with norm bigger than γ/C', for every t in the subinterval. This allows to find global small perturbations of $s_{t,k}$ in every $[t_i, t_{i+1}]$. Reducing α and enlarging C' we may suppose that the same happens to any perturbation of the original $s_{t,k}$ at C^1-distance at most α.

Now work as follows. For the first subinterval, consider $s_{t,k}^1 = s_{t,k}$, $t \in [0, t_1]$. We find a perturbation $\sigma_{t,k}^1$, $t \in [0, t_1]$, such that $|s_{t,k}^1 - \sigma_{t,k}^1| < \alpha/2$ and $\sigma_{t,k}^1$ is η_1-generic of order $l+1$, for some $\eta_1 > 0$. Set $\sigma_{t,k} = \sigma_{t,k}^1$ for $t \in [0, t_1]$. In the second subinterval, perturb $s_{t,k}^2 = s_{t,k}^1 + (\sigma_{t_1,k}^1 - s_{t_1,k}^1)$, $t \in [t_1, t_2]$, to find $\sigma_{t,k}^2$ satisfying $|s_{t,k}^2 - \sigma_{t,k}^2| < \alpha/4$ and $\sigma_{t,k}^2$ is η_2-generic of order $l+1$, for some $\eta_2 > 0$. To glue this perturbation with the previous one puts

$$\sigma_{t,k} = \begin{cases} s_{t,k}^2 + \frac{t-t_1}{\epsilon}(\sigma_{t,k}^2 - s_{t,k}^2), & t \in [t_1, t_1+\epsilon] \\ \sigma_{t,k}^2, & t \in [t_1+\epsilon, t_2]. \end{cases}$$

Here $\epsilon > 0$ is chosen so small that $|s_{t,k}^2 - \sigma_{t_1,k}^1|_{C^1} < \rho/2$, for $t \in [t_1, t_1+\epsilon]$, and we require also that the perturbation satisfies $|s_{t,k}^2 - \sigma_{t,k}^2| < \rho/2$, where $\rho > 0$ is a number such that any perturbation of $\sigma_{t_1,k}^1$ of C^1-norm less than ρ is $\eta_1/2$-generic of order $l+1$. This defines $\sigma_{t,k}$ for $t \in [0, t_2]$ already.

Proceeding in this way we finally find $\sigma_{t,k}$, $t \in [0,1]$, which is η-generic of order $l+1$, for some $\eta > 0$, with $|\sigma_{t,k} - s_{t,k}| < \alpha$. □

2.5. **Lifting asymptotically holomorphic embeddings.** In this Subsection we aim to prove that the sequences of asymptotically holomorphic embeddings into $\mathbb{CP}^{2n+1}$ that we are considering in Theorem 1.2 come always from asymptotically holomorphic sequences of sections s_k of $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ which are γ-projectizable and γ-generic of order n, for some $\gamma > 0$ (at least for k large).

Proof of Lemma 2.15. Suppose that we have a sequence of γ-asymptotically holomorphic embeddings $\phi_k : M \to \mathbb{CP}^{2n+1}$, for some $\gamma > 0$, with $\phi_k^*\mathcal{U} = L^{\otimes k}$. Here $\mathcal{U}$ is the hyperplane line bundle defined over the projective space. The dual of $\mathcal{U}$ is the universal line bundle

$$\mathcal{E} = \{(l,s) \big| s \in l\} \subset \mathbb{CP}^{2n+1} \times \mathbb{C}^{2n+2} = \underline{\mathbb{C}}^{2n+2},$$

interpreted as a sub-bundle of the trivial bundle $\underline{\mathbb{C}}^{2n+2}$.

Consider the following sequence of line bundles, $E_k = \phi_k^*\mathcal{E} \otimes L^{\otimes k} = \underline{\mathbb{C}} \subset \mathbb{C}^{2n+2} \otimes L^{\otimes k}$, which are topologically trivial. We look for everywhere non-zero sections s_k of $E_k \subset \mathbb{C}^{2n+2} \otimes L^{\otimes k}$ as they satisfy $\phi_k = \mathbb{P}(s_k)$.

Let $\mathscr{P}(\epsilon, x)$ be the C^1-open property for sequences of sections s_k of E_k of being ϵ-transverse to 0 at the point x (see Definition 2.2). We shall use Proposition 2.10 to find sequences of sections s_k which are η-transverse to 0, for some $\eta > 0$. Fix

any asymptotically holomorphic sequence s_k of E_k (e.g. the zero sections) which will act as the starting point of our perturbation process. Let $x \in M$. Consider the sections $s_{k,x}^{\mathrm{ref}}$ of $L^{\otimes k}$ given by Lemma 2.5 and define also the local sections of the line bundle $\phi_k^* \mathcal{E} \subset \underline{\mathbb{C}}^{2n+2}$,

$$\sigma_k : B_{g_k}(x, c) \to \mathbb{C}^{2n+2},$$

by setting $\sigma_k(x)$ any vector of norm 1 in the direction defined by $\phi_k(x)$ and satisfying the condition $\nabla_r \sigma_k(y) \perp \sigma_k(y)$, for any $y \in B_{g_k}(x, c)$, where r is the radial vector field from x. This determines σ_k uniquely. The following estimates hold:

$$\begin{aligned} &|\sigma_k(y)| = 1, \ |\nabla \sigma_k(y)| = O(1 + d_k(x, y)), \\ &|\bar{\partial} \sigma_k(y)| = O(k^{-1/2}(1 + d_k(x, y))), \ |\nabla \bar{\partial} \sigma_k(y)| = O(k^{-1/2}(1 + d_k(x, y))). \end{aligned} \tag{5}$$

The first one follows from $\nabla_r \langle \sigma_k, \sigma_k \rangle = \langle \nabla_r \sigma_k, \sigma_k \rangle + \langle \sigma_k, \nabla_r \sigma_k \rangle = 0$. For the second one, write $\nabla \sigma_k = \nabla \phi_k + \langle \nabla \sigma_k, \sigma_k \rangle \sigma_k$, where we identify $T_{\phi_k(y)} \mathbb{CP}^{2n+1} = [\sigma_k(y)]^{\perp} \subset \mathbb{C}^{2n+2}$, isometrically. We already know that $|\nabla \phi_k| = O(1)$. So

$$\begin{aligned} \nabla_r \langle \nabla \sigma_k, \sigma_k \rangle &= \langle \nabla \nabla_r \sigma_k, \sigma_k \rangle + \langle \nabla \sigma_k, \nabla_r \sigma_k \rangle = \\ &= \langle \nabla \nabla_r \sigma_k, \sigma_k \rangle + \langle \nabla \sigma_k, \nabla_r \sigma_k \rangle = \\ &= -\langle \nabla_r \sigma_k, \nabla \sigma_k \rangle + \langle \nabla \sigma_k, \nabla_r \sigma_k \rangle = \\ &= -\langle \nabla_r \phi_k, \nabla \phi_k \rangle + \langle \nabla \phi_k, \nabla_r \phi_k \rangle = O(1), \end{aligned}$$

The first equality uses that ∇ is the standard derivative for functions with values in $\mathbb{C}^{2n+2}$, and hence the second derivatives commute. The second equality follows from $\langle \nabla_r \sigma_k, \sigma_k \rangle = 0$. So we have that $\langle \nabla \sigma_k, \sigma_k \rangle = O(d_k(x, y))$ and hence $|\nabla \sigma_k| = O(1 + d_k(x, y))$. The other two cases are worked out analogously.

Now define the application

$$f_k = \frac{s_k}{s_{k,x}^{\mathrm{ref}} \sigma_k} : B_{g_k}(x, c) \to \mathbb{C},$$

which is asymptotically holomorphic by construction. Using a complex Darboux chart we trivialize $B_{g_k}(x, c)$ to obtain (scaling the coordinates by appropiate universal constants) an application $\hat{f}_k : B^+ \to \mathbb{C}$ to which we apply Proposition 2.10 to obtain $w_k \in B(0, \delta)$ such that $\hat{f}_k - w_k$ is η-transverse to 0 in B, where $\eta = \delta(\log(\delta^{-1}))^{-p}$. Rescaling and passing to the manifold, we have that $f_k - C w_k$ is $C' \eta$-transverse to 0, for some universal constants C and C'. Define the sequence of sections $\tau_{k,x} = -w_k s_{k,x}^{\mathrm{ref}} \sigma_k$ of E_k, which is asymptotically holomorphic and has Gaussian decay by (5), to get a perturbation satisfying the conditions in Proposition 2.8.

Thus there exists an asymptotically holomorphic sequence s_k of sections of E_k which is η-transverse to 0, for some $\eta > 0$. For k large enough, the zeroes of s_k is a symplectic submanifold representing the trivial homology class, hence the empty set. So s_k is nowhere vanishing and hence $\phi_k = \mathbb{P}(s_k)$.

We have that s_k is an asymptotically holomorphic sequence of sections of $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$. Let us check that s_k is η-projectizable, i.e. that $|s_k| \geq \eta$ everywhere. Suppose that this is not the case and take the point $x \in M$ where $|s_k|$ attains its minimum. As $|s_k(x)| < \eta$, η-transversality implies that $|\nabla s_k(x)| \geq \eta$. Also s_k is asymptotically holomorphic, so for k large $\nabla s_k(x) : T_x M \to (E_k)_x$ is surjective. Take $v \in T_x M$ such that $\nabla_v s_k(x) = s_k(x)$. Evaluating the equality

$$\nabla |s_k|^2 = \langle \nabla s_k, s_k \rangle + \langle s_k, \nabla s_k \rangle.$$

at the point x and along the direction of v, we obtain $|s_k(x)|^2 = 0$, which is impossible since we have already proved that s_k is nowhere vanishing.

Finally the extension to the one-parameter case is trivial. □

3. Estimated intersections of symplectic submanifolds

3.1. **Notions on estimated euclidean geometry.** In order to set up the definitions needed in Subsection 3.2 we state the relevant notions and results on angles between subspaces of euclidean spaces that we shall need. From now on we assume that we are in $\mathbb{R}^n$ equipped with the standard euclidean inner product, but all the proofs apply to a general finite dimensional euclidean space.

The angle between two non-zero vectors $v, w \in \mathbb{R}^n$ is defined as

$$\angle(v,w) = \arccos\left(\frac{\langle v,w\rangle}{|v||w|}\right) \in [0,\pi].$$

The angle is symmetric and satisfies the classical triangular inequality,

$$\angle(u,w) \leq \angle(u,v) + \angle(v,w),$$

for non-zero vectors $u, v, w \in \mathbb{R}^n$. Also the angle of a vector $u \neq 0$ respect to a subspace $V \neq \{0\}$ is defined as

$$\angle(u,V) = \min_{v\in V-\{0\}}\{\angle(u,v)\} = \angle(u,v(u)) \in [0,\frac{\pi}{2}],$$

where $v : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto V, well understood that when $v(u) = 0$ the angle is $\pi/2$.

Definition 3.1. *The maximum angle of a subspace $U \neq \{0\}$ with respect to a subspace $V \neq \{0\}$ is defined as*

$$\angle_M(U,V) = \max_{u\in U-\{0\}} \angle(u,V).$$

Notice that this angle is not in general symmetric. But in the case $\dim U = \dim V$ symmetry does hold. This is easily checked by constructing an orthogonal transformation permuting the two subspaces. Indeed the maximum angle $\angle_M(U,V)$ gives a notion of proximity between U and V whenever $\dim U \leq \dim V$.

Lemma 3.2. *Given U, V, W non zero-subspaces in $\mathbb{R}^n$ then:*

$$\angle_M(U,W) \leq \angle_M(U,V) + \angle_M(V,W).$$

Proof. We will denote by $v(u)$ the orthogonal projection of the vector u onto the subspace V. In the following inequalities, if $v(u) = 0$, we suppose that the angle in which this expression appears is $\pi/2$. We have

$$\begin{aligned}
\angle_M(U,W) &= \max_{u\in U-\{0\}}\{\min_{w\in W-\{0\}}\{\angle(u,w)\}\} \leq \\
&\leq \max_{u\in U-\{0\}}\{\min_{w\in W-\{0\}}\{\angle(u,v(u)) + \angle(v(u),w)\}\} = \\
&= \max_{u\in U-\{0\}}\{\angle(u,v(u)) + \min_{w\in W-\{0\}}\{\angle(v(u),w)\}\} \leq \\
&\leq \max_{u\in U-\{0\}}\{\angle(u,v(u))\} + \max_{u\in U-\{0\}}\{\min_{w\in W-\{0\}}\{\angle(v(u),w)\}\} \leq \\
&\leq \angle_M(U,V) + \max_{v\in V-\{0\}}\{\min_{w\in W-\{0\}}\{\angle(v,w)\}\} \leq \\
&\leq \angle_M(U,V) + \angle_M(V,W).
\end{aligned}$$

□

Definition 3.3. *The minimum angle between two non-zero subspaces U, V of $\mathbb{R}^n$ is defined as follows:*

- *If $\dim U + \dim V < n$ then $\angle_m(U,V) = 0$.*
- *If their intersection is not transversal then $\angle_m(U,V) = 0$.*
- *If their intersection is transversal then let W be their intersection. Define U_c as the orthogonal subspace in U to W, and V_c in the same way. Then $\angle_m(U,V) = \min_{u\in U_c-\{0\}}\{\angle(u,V_c)\} \in [0,\pi/2]$.*

The definition is symmetric because (in the transversal case)

$$\angle_m(U,V) = \min_{u\in U_c-\{0\}}\{\min_{v\in V_c-\{0\}}\{\angle(u,v)\}\}$$

and the two minima commute. Also $\angle_m(U,V) = \min_{u\in U_c-\{0\}}\{\angle(u,V)\}$.

Lemma 3.4. *For non-zero subspaces U and V of $\mathbb{R}^n$ we have that*

$$\angle_m(U,V) = \min_{u\in U^\perp-\{0\}}\{\min_{v\in V^\perp-\{0\}}\{\angle(u,v)\}\}$$

Proof. This is trivial in the case $\dim U + \dim V < n$ or when U and V do not intersect transversely. In the transversal case, we can restrict ourselves to the subspace $(U\bigcap V)^\perp$ to compute the angles. So without loss of generality we can suppose that $U\oplus V = \mathbb{R}^n$, $U_c = U$ and $V_c = V$. As $\dim U = \dim V^\perp$, we may construct an orthogonal transformation ϕ permuting U and $V^\perp$, i.e. $\phi(U) = V^\perp$ and $\phi(V^\perp) = U$. Therefore also $\phi(V) = U^\perp$. So

$$\angle_m(U,V) = \angle_m(\phi(U),\phi(V)) = \angle_m(V^\perp,U^\perp) = \min_{u\in U^\perp-\{0\}}\{\min_{v\in V^\perp-\{0\}}\{\angle(u,v)\}\},$$

which proves the lemma. □

Proposition 3.5. *For non-zero subspaces U,V,W of $\mathbb{R}^n$ we have that*

$$\angle_m(U,V) \le \angle_M(U,W) + \angle_m(W,V).$$

Proof. By Lemma 3.4 we have that

$$\begin{aligned}\angle_m(U,V) &= \min_{u\in U^\perp-\{0\}}\{\min_{v\in V^\perp-\{0\}}\{\angle(u,v)\}\} \le \\ &\le \min_{u\in U^\perp-\{0\}}\{\angle(u,w)\} + \min_{v\in V^\perp-\{0\}}\{\angle(w,v)\},\end{aligned}$$

for any $w\in\mathbb{R}^n$. Choose $w_0\in W^\perp-\{0\}$ satisfying

$$\angle_m(W,V) = \min_{w\in W^\perp-\{0\}}\{\min_{v\in V^\perp-\{0\}}\{\angle(w,v)\}\} = \min_{v\in V^\perp-\{0\}}\{\angle(w_0,v)\}.$$

Then we have

$$\angle_m(U,V) \le \min_{u\in U^\perp-\{0\}}\{\angle(u,w_0)\} + \angle_m(W,V) \le \angle_M(W^\perp,U^\perp) + \angle_m(W,V).$$

The result follows once we show that $\angle_M(W^\perp,U^\perp) = \angle_M(U,W)$. Put $\angle_M(U,W) = \alpha$. Let $u\in U$ with $\angle(u,W) = \alpha$. Denoting by w the projection of u onto $W^\perp$, we have that $\angle(u,W^\perp) = \angle(u,w) = \frac{\pi}{2}-\alpha$. So $\angle(w,U) \le \frac{\pi}{2}-\alpha$ and hence $\angle(w,U^\perp)\ge\alpha$. This implies that $\angle_M(W^\perp,U^\perp)\ge\alpha = \angle_M(U,W)$. The opposite inequality follows by symmetry. □

Corollary 3.6. *Given non-zero subspaces U,U',V of $\mathbb{R}^n$ with $\angle_m(U,V) > \epsilon$ and $\angle_M(U,U') < \delta$ then $\angle_m(U',V) > \epsilon - C\delta$, where C is a universal constant ($C = 1$ in fact).* □

The following result will be very important for our purposes.

Proposition 3.7. *Given $\epsilon > 0$ and $U\in \mathrm{Gr}(m,n)$, $V\in\mathrm{Gr}(r,n)$ subspaces verifying that $\angle_m(U,V) > \epsilon$. Then there are $\gamma_0 > 0$ and a constant C, depending only on ϵ, such that for any $\gamma < \gamma_0$, if $U'\in Gr(m,n)$ and $V'\in Gr(r,n)$ verify that*

$$\angle_M(U,U') < \gamma,\ \angle_M(V,V') < \gamma,$$

then U' and V' intersect transversally and $\angle_M(U\bigcap V, U'\bigcap V') < C\gamma$.

Proof. By Proposition 3.5 choosing $\gamma_0 > 0$ small enough, only depending on ϵ, we can assure that the following intersections are transversal $U \bigcap V = W$, $U \bigcap V'$, $U' \bigcap V$ and $U' \bigcap V' = W'$ and that $\angle_m(U', V') \geq \epsilon/2$. By Lemma 3.2 we have

$$\angle_M(W, W') \leq \angle_M(W, U \bigcap V') + \angle_M(W', U \bigcap V').$$

We are going to bound the first term in the right hand side of the inequality, the bounding of the second term being analogous.

Put $s = \dim W = r + m - n$. Choose an orthonormal basis $(e_1, \ldots, e_s)$ of W, extend it to an orthonormal basis $(e_1, \ldots, e_r)$ of V and finally extend it to an orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$. Note that $(e_{s+1}, \ldots, e_r)$ is an orthonormal basis of V_c. As $\angle_m(U, V) = \angle_m(U_c, V) > \epsilon$ and $\angle_M(V, V') < \gamma_0$ we have $\angle_m(U_c, V') > \epsilon/2$ (decreasing γ_0 if necessary). So $U_c \cap V' = \{0\}$. Recalling that $V \oplus U_c = \mathbb{R}^n$, we see that there is a basis $(e_1 + \varepsilon_1, \ldots, e_r + \varepsilon_r)$ for V' where $\varepsilon_j \in U_c$. Using that $\angle_m(U, V) > \epsilon$ and that the decomposition $\mathbb{R}^n = W \oplus V_c \oplus V^\perp$ is orthogonal, we have

$$\begin{aligned} \mathrm{pr}^\perp_W(\varepsilon_j) &= 0, \\ \mathrm{pr}^\perp_{V_c}(\varepsilon_j) &\leq |\cos\epsilon||\varepsilon_j|, \\ \mathrm{pr}^\perp_{V^\perp}(\varepsilon_j) &\geq \sqrt{1 - |\cos\epsilon|^2}|\varepsilon_j| = |\sin\epsilon||\varepsilon_j|. \end{aligned}$$

Checking the angle of $e_j + \varepsilon_j$ with respect to V, we get that

$$\angle_M(V, V') \geq \arctan \frac{|\sin\epsilon||\varepsilon_j|}{1 + |\cos\epsilon||\varepsilon_j|} \geq \arctan\left(\frac{\sin\epsilon}{1 + |\varepsilon_j|}|\varepsilon_j|\right). \tag{6}$$

For $\gamma_0 < \arctan \frac{\sin\epsilon}{2}$, (6) implies that $|\varepsilon_j| < 1$ and hence we get that $\angle_M(V, V') \geq \arctan(\frac{\sin\epsilon}{2}|\varepsilon_j|) \geq \frac{4}{\pi}\frac{\sin\epsilon}{2}|\varepsilon_j|$, or said otherwise $|\varepsilon_j| < C\angle_M(V, V')$ for a constant C depending on ϵ.

Now let us compute $\angle_M(W, U \bigcap V')$. The intersection $U \bigcap V'$ has basis $(e_1 + \varepsilon_1, \ldots, e_s + \varepsilon_s)$. Take a general vector $u = \sum_{i=1}^s a_i(e_i + \varepsilon_i)$ in $U \bigcap V'$ and compute $\angle(u, W)$. We may suppose that $a = (a_1, \ldots, a_s)$ has norm one. Write $\varepsilon = \sum_{i=1}^s a_i\varepsilon_i$. Then

$$\angle(u, W) = \arccos \frac{1}{\sqrt{1 + |\varepsilon|^2}} = \arctan|\varepsilon| \leq |\varepsilon|.$$

Finally

$$\angle_M(W, U \bigcap V') \leq \max_{|a|=1} |\sum_{i=1}^s a_i\varepsilon_i| = \max_{1 \leq i \leq s} |\varepsilon_i| \leq C\angle_M(V, V') \leq C\gamma.$$

□

Now we are going to set up the relationship between the transversality of maps in the Donaldson-Auroux approach and the angles defined above. This is the content of the following

Lemma 3.8. *Let U, V be two non-zero subspaces of $\mathbb{R}^n$ and let $g : U \to V$ and $h : U \to V^\perp$ be the projections from U with respect to the decomposition $\mathbb{R}^n = V \oplus V^\perp$. If h has a right inverse θ satisfying $|\theta| < \gamma^{-1}$ for some $\gamma > 0$ then $\angle_m(U, V) > \gamma$.*

Proof. In the first place, as h is onto, the intersection between U and V is transversal. Let $W = U \cap V$. Define $\hat\theta = \mathrm{pr}^\perp_{U_c} \circ \theta : V^\perp \to U_c$, which is an inverse of $h : U_c \to V^\perp$ such that $|\hat\theta| < \gamma^{-1}$. Now consider any $u \in U_c - \{0\}$ and put $v = h(u)$. Then

$$\angle(u, V) = \arcsin \frac{|h(u)|}{|u|} = \arcsin \frac{|v|}{|\hat\theta(v)|} > \arcsin \frac{1}{\gamma^{-1}} > \gamma,$$

and the proof is concluded. □

3.2. **Projective symplectic geometry.** In this Subsection we will prove Theorem 1.3. This will provide a geometric proof of Bertini's theorem, the main result of [Do96]. Although our proof is more technical and long, it has the advantage of giving us a more general kind of symplectic submanifolds than those in [Do96, Au97]. In fact our technique will allow us a simple generalization to solve the problem of constructing determinantal symplectic submanifolds in Section 5. First of all, in order to measure the holomorphicity of submanifolds, let us introduce the complex angle of even dimensional subspaces $V \subset \mathbb{C}^n$ as

$$\begin{aligned} \beta : \mathrm{Gr}_{\mathbb{R}}(2r, 2n) &\to [0, \pi/2] \\ V &\to \angle_M(V, JV). \end{aligned}$$

Clearly $\beta(V) = 0$ if and only if V is complex and $\beta(V) < \pi/2$ if and only if V is symplectic.

Definition 3.9. *Let (M, ω) be a symplectic submanifold endowed with a compatible almost complex structure J. A sequence of submanifolds $S_k \subset M$ is asymptotically holomorphic if $\beta(TS_k) = O(k^{-1/2})$.*

Note that if S_k are asymptotically holomorphic submanifolds then they are symplectic for k large. If $\phi_k : M \to \mathbb{CP}^N$ is a sequence of asymptotically holomorphic embeddings then $\phi_k(M)$ is a sequence of asymptotically holomorphic submanifolds.

Proposition 3.10. *Let $\phi_k^1 : (M_1, J_1) \to \mathbb{CP}^N$ and $\phi_k^2 : (M_2, J_2) \to \mathbb{CP}^N$ be two sequences of asymptotically holomorphic embeddings. Suppose that there exists $\epsilon > 0$ independent of k such that for any $x \in \phi_k^1(M_1) \bigcap \phi_k^2(M_2)$, the minimum angle between $(\phi_k^1)_* TM_1(x)$ and $(\phi_k^2)_* TM_2(x)$ is greater than ϵ. Then $S_k = \phi_k^1(M_1) \bigcap \phi_k^2(M_2)$ is a sequence of asymptotically holomorphic submanifolds (hence symplectic for k large). Also $S_k^j = (\phi_k^j)^{-1}(S_k)$ is a sequence of asymptotically holomorphic submanifolds of M_j, $j = 1, 2$. Moreover there exists a sequence of compatible almost complex structures J_k^j of M_j such that S_k^j is pseudoholomorphic for J_k^j, $|J_k^j - J_j| = O(k^{-1/2})$ and ϕ_k^j restricted to (S_k^j, J_k^j) is a sequence of asymptotically holomorphic embeddings in $\mathbb{CP}^N$, $j = 1, 2$.*

The same statement holds for the case of one-parameter families of embeddings $(\phi_{t,k}^1)_{t \in [0,1]}$ and $(\phi_{t,k}^2)_{t \in [0,1]}$.

Remark that M_1 and M_2 are not necessarily compact manifolds.

Proof. Let J_0 be the standard complex structure of $\mathbb{CP}^{2n+1}$. Then $\angle_M((\phi_k^j)_* TM, J_0(\phi_k^j)_* TM) = O(k^{-1/2})$ for $j = 1, 2$. By Proposition 3.7, $\angle_M(TS_k, J_0 TS_k) = O(k^{-1/2})$. As $|(\phi_k^j)_* J_j - J_0| = O(k^{-1/2})$ on $(\phi_k^j)_* TM$, we have $\angle_M(TS_k, (\phi_k^j)_* J_j TS_k) = O(k^{-1/2})$ and so $\angle_M(TS_k^j, J_j TS_k^j) = O(k^{-1/2})$, i.e. S_k^j is a sequence of asymptotically holomorphic submanifolds of M_j.

Finally we have to build J_k^j on M_j such that $|J_k^j - J_j| = O(k^{-1/2})$ and S_k^j is J_k^j-holomorphic. Take the composition $\tilde{J}_k^j : TS_k^j \subset TM \xrightarrow{J_j} TM \overset{\mathrm{pr}^\perp}{\twoheadrightarrow} TS_k^j$ with square close to -1, for k large enough. So we can homotop it to an almost complex structure J_k^j on S_k^j. Then we extend this J_k^j to a small tubular neighborhood of S_k^j by giving a complex structure to the normal bundle of S_k^j. Finally a homotopy between J_k^j and J_j allows us to extend J_k^j off a little bigger neighborhood of S_k^j matching with J_j on the border. This gives the required J_k^j.

The result for continuous one-parameter families is trivial from the non-parametric case. □

Let us have a smooth submanifold N of a manifold X. If we fix a metric on X we can define a geodesic flow φ_t. In particular, following the perperdicular directions to N we can identify a tubular neighborhood of the zero section of the normal

bundle of N (defined as $|n| < t_0$, $n \in \nu(N)$, for some small $t_0 > 0$) with a tubular neighborhood $U_N \subset X$ of N. So we can define an integrable distribution D_N in U_N as

$$D_N(\varphi_n(x)) = (\varphi_n)_* T_x N, \ \forall x \in N, n \in \nu(N), |n| < t_0.$$

where $(\varphi_n)_*$ denotes parallel transport along the geodesic tangent to n.

Definition 3.11. *Suppose $\phi_k : M \to X$ is a sequence of asymptotically holomorphic embeddings into a Hodge manifold X. Let us fix a complex submanifold $N \subset X$. We say that ϕ_k is σ-transverse to N, with $\sigma < t_0$, if for all $x \in M$ and all k,*

$$d(\phi_k(x), N) < \sigma \Rightarrow \angle_m((\phi_k)_*(T_x M), D_N(\phi_k(x))) > \sigma.$$

This property is C^1-open, i.e. given ϕ_k an embedding η-transverse to N, then a perturbation of $\hat{\phi}_k$ with $d_{C^1}(\phi_k, \hat{\phi}_k) < \delta$ is $(\eta - C\delta)$-transverse to N, where C is a universal constant.

Obviously a σ-transverse sequence of embeddings ϕ_k verifies the conditions of Proposition 3.10 with $\phi_k^1 = \phi_k : M \to X$ and $\phi_k^2 = i : N \hookrightarrow X$. The following result then completes the proof of Theorem 1.3

Theorem 3.12. *Let $\phi_k = \mathbb{P}(s_k)$, where s_k is an asymptotically holomorphic sequence of sections of $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ which is γ-projectizable and γ-generic of order n, for some $\gamma > 0$. Let us fix a holomorphic submanifold N in $\mathbb{CP}^{2n+1}$. Then for any $\delta > 0$ there exists an asymptotically holomorphic sequence of sections σ_k of $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ such that*

1. $|\sigma_k - s_k|_{g_k, C^1} < \delta$.
2. *$\hat{\phi}_k = \mathbb{P}(\sigma_k)$ is a η-asymptotically holomorphic embedding in $\mathbb{CP}^{2n+1}$ which is ϵ-transverse to N, for some $\eta > 0$ and $\epsilon > 0$. In the case* $\dim M + \dim N < 2n+1$ *we actually have that $d_{FS}(\hat{\phi}_k(M), N)) > \epsilon$, for k large enough.*

Moreover the result can be extended to one-parameter continuous families of complex submanifolds $(N_t)_{t \in [0,1]}$, taking in this case as starting point a continuous family $\phi_{t,k} = \mathbb{P}(s_{t,k})$ where $s_{t,k}$ are asymptotically J_t-holomorphic sections of $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ which are γ-projectizable and γ-generic of order n, for some $\gamma > 0$.

The proof of this result will be the content of Subsection 3.3. Now we shall extract some corollaries from it. The first one is the main Theorem of [Do96].

Corollary 3.13. *Given a compact symplectic manifold (M, ω), suppose that $[\omega/2\pi] \in H^2(M, \mathbb{R})$ is the reduction of an integral class h. Then for k large enough there exists symplectic submanifolds realizing the Poincaré dual of kh. Moreover, perhaps by increasing k, we can assure that all the symplectic submanifolds realizing this Poincaré dual, constructed as transverse intersections with a fixed complex hyperplane of asymptotically holomorphic sequences of embeddings with respect to two compatible almost complex structures, are isotopic. The isotopy can be made by symplectomorphisms.*

Recall that we obtain an isotopy result similar to [Au97], where the isotopy of the submanifolds obtained as zero sets of a special set of sections of the line bundle $L^{\otimes k}$ is obtained. The Auroux' more general case of vector bundles will be proved in Section 4.

Proof. The existence result is a direct consequence of the previous statements. By Theorem 2.11 we build an asymptotically holomorphic sequence of embeddings to $\mathbb{CP}^{2n+1}$. In $\mathbb{CP}^{2n+1}$ we choose a complex hyperplane H. By Theorem 3.12 we perturb the sequence of embeddings to find a new asymptotically holomorphic sequence of embeddings ϕ_k such that $\phi_k(M)$ intersects H with minimum angle greater than $\epsilon > 0$. Finally using Proposition 3.10 we obtain that $\phi_k(M) \bigcap H = H_M$ is an asymptotically holomorphic sequence of submanifolds, and these manifolds are

symplectic for k large enough. Also $\phi_k^{-1}(H_M)$ is a symplectic submanifold of M for k large enough. A direct topological argument shows us that it is Poincaré dual of kh.

For the isotopy statement, let us assume that there are two sequences of symplectic submanifolds W_k^0 and W_k^1, both Poincaré dual of kh, obtained as intersections between two η-asymptotically J_j-holomorphic sequences $\mathbb{P}(s_{k,j})$, $j=0,1$, and two fixed complex hyperplanes H_0 and H_1 in $\mathbb{CP}^{2n+1}$ with angles greater than a fixed $\epsilon>0$. Then we will prove that in this case they are isotopic. We only have to construct the straight segment H_t, in the dual space, of hyperplanes connecting H_0 and H_1. Also we define the following family of asymptotically holomorphic sequences:

$$s_{t,k} = \begin{cases} (1-3t)s_{0,k}, & \text{with } J_t = J_0, \qquad t \in [0,1/3] \\ 0, & \text{with } J_t = \mathrm{Path}(J_0, J_1), \qquad t \in [1/3, 2/3] \\ (3t-2)s_{1,k}, & \text{with } J_t = J_1, \qquad t \in [2/3, 1]. \end{cases}$$

By means of Theorem 3.12, we obtain a family $\phi_{t,k} = \mathbb{P}(\sigma_{t,k})$ of asymptotically J_t-holomorphic embeddings which are $\eta/2$-transverse to N, choosing the perturbation $\delta>0$ in the statement of the theorem, in such a way that

$$\eta - C\delta > \eta/2, \tag{7}$$

where C is the universal constant of the C^1-openness of the transversality to N. This gives us a family of symplectic isotopic submanifolds $(W_k^t)'$ in M for each fixed large k. The problem is that W_k^0 does not coincide with $(W_k^0)'$ (and respectively for $t=1$). Using (7) we can assure that they are isotopic, in fact the linear segment $((1-t)\sigma_{0,k} + ts_{0,k})_{t\in[0,1]}$ provides a family of asymptotically holomorphic embeddings transverse to H_0, for k large enough giving the desired isotopy. □

The constructive technique of Theorem 3.12 is more general because we do not have to choose hyperplanes in $\mathbb{CP}^{2n+1}$ to make the intersection. However, the difficulty in finding topological information about the constructed submanifolds makes that we cannot assure that they are more general that the ones produced in [Au97]. To overcome this problem we are going to construct in Section 5 a special kind of submanifolds where we can compute symplectic invariants using similar results from algebraic geometry.

3.3. **Estimated intersections in $\mathbb{CP}^{2n+1}$.** Now we aim to prove Theorem 3.12. Our objective is to find sequences ϕ_k of asymptotically holomorphic embeddings which are σ-transverse to N.

Proof of Theorem 3.12. As usual we begin with the simplest case, when the complex codimension of N is 1. Also we consider the non-parametric case, being the parametric one a simple generalization. We say that a sequence of sections s_k which is $\gamma/2$-projectizable and $\gamma/2$-generic of order n verifies $\mathscr{P}(\epsilon, x)$ if $\mathbb{P}(s_k)$ is ϵ-transverse to N at the point x. This property is local and open in C^1-sense, for $\epsilon < t_0$. To make use of Proposition 2.8 we need to find local sections with Gaussian decay obtaining local transversality. To achieve this local transversality we are going to use Proposition 2.10. (We could have used instead the case $m=1$ proved in [Do96, Au97], by increasing a little the complications of the globalization process, which is the way followed by Auroux in [Au97, Au99].)

As N is a fixed holomorphic submanifold, we may fix a finite covering of $\mathbb{CP}^{2n+1}$ by balls U_j such that N is defined as the zero set of a holomorphic function $f_j : U_j \to \mathbb{C}$ in each U_j and such that for any $z_1, z_2 \in U_j \cap U_N$, $\angle_M(D_N(z_1), D_N(z_2)) \le \varepsilon$, and for any $z_1, z_2 \in U_j$, $\angle_M(\ker df_j(z_1), \ker df_j(z_2)) \le \varepsilon$, with $\varepsilon > 0$ an arbitrarily small number fixed along the proof.

We choose a constant C independent of k such that $|\nabla \phi_k|_{g_k} \leq C$. Therefore $\phi_k(B_{g_k}(x,c)) \subset B_{g_{FS}}(\phi_k(x), Cc)$, for any c. Now we choose $c > 0$ small enough satisfying the following premises:

1. Let $x \in M$. With a transformation of $U(2n+2)$ in $\mathbb{C}^{2n+2}$, we may suppose that $s_k(x) = (s_k^0(x), 0, \dots, 0)$. As s_k is γ-projectizable and asymptotically holomorphic, we can choose a universal g_k-radius c with $|s_k^0| \geq \gamma/2$ on $B_{g_k}(x, 20c)$. Also the sections $s_{k,x}^{\mathrm{ref}}$ of Lemma 2.5 satisfy $|s_{k,x}^{\mathrm{ref}}| \geq c_s$ on $B_{g_k}(x, 20c)$. Note that $\phi_k(B_{g_k}(x, 20c)) \subset B_{g_{FS}}(\phi_k(x), 20Cc)$.
2. We use the standard chart Φ_0 for $\mathbb{CP}^{2n+1}$ around $p = \phi_k(x) = [1, 0, \dots, 0]$ to trivialize the ball $B_{g_{FS}}(p, 20Cc)$. We may choose c small enough so that Φ_0 is near an isometry, in the sense that
$$\frac{2}{3}|\Phi_0(q)| \leq d_{FS}(p,q) \leq 2|\Phi_0(q)|.$$
for $q \in B_{g_{FS}}(p, 20Cc)$. Also we require $|\nabla \Phi_0| \leq 2$ in such ball. With respect to this trivialization the map ϕ_k is given locally as
$$\begin{array}{rcl} f_k = \Phi_0 \circ \phi_k : B_{g_k}(x, 20c) & \to & B(0, 40Cc) \\ y & \mapsto & \left(\dfrac{s_k^1(y)}{s_k^0(y)}, \dots, \dfrac{s_k^{2n+1}(y)}{s_k^0(y)} \right). \end{array}$$
Clearly $|\nabla f_k| \leq 2C$ uniformly in k.
3. We can reduce c so that, for any p, $B_{g_{FS}}(p, 20Cc) \subset U_j$ for some U_j. Therefore N is defined in $B(0, 15Cc)$ by a function $f : B(0, 15Cc) \to \mathbb{C}$. Call $Z = Z(f)$ in such ball. The angle condition means that $\ker df(z_1)$, $\ker df(z_2)$ are close enough (say less than $\pi/6$) for $z_1, z_2 \in Z$.

Let $x \in M$. In the case $d(\phi_k(x), N) \geq 2Cc$, as we perform a small perturbation, say of norm $\delta > 0$ such that $d_{FS}(\phi_k(x), \hat{\phi}_k(x)) < \frac{1}{2}Cc$, for all $x \in M$, there is still $\frac{1}{2}Cc$-transversality at a c-neighbourhood of x. So we are finished.

Suppose $d(\phi_k(x), N) < 2Cc$. Then take a point $z_0 \in B(0, 4Cc) \cap Z$ which gives the minimum distance from 0 to Z. If $0 \notin Z$, take $v = (v_1, \dots, v_{2n+1}) \in \mathbb{C}^{2n+1}$ a unitary vector in the direction of the complex line from 0 to z_0. This vector is perpendicular to $T_{z_0}Z$. If $0 \in Z$ then let v be a unitary vector orthogonal to $T_0 Z$. Therefore

$$\langle df(z), v \rangle \geq \frac{1}{2}|df(z)| \tag{8}$$

for any $z \in Z \cap B(0, 15Cc)$, by the condition on the angle (taking $\epsilon > 0$ small enough).

Let $r_0 \in \mathbb{C}$ with $r_0 v = z_0 \in Z$. We look for a function $r_k = r_k(y) : B_{g_k}(x,c) \to \mathbb{C}$ such that $r_k(x) = r_0$ and

$$f\left(f_k^1(y) + r_k v_1, \dots, f_k^{2n+1}(y) + r_k v_{2n+1}\right) = 0. \tag{9}$$

This corresponds to tracing a straight line from the image of the point $y \in B_{g_k}(x,c)$ to Z with direction v. Such r_k can be found with the use of the implicit function theorem applied to the function $F : B_{g_k}(x,c) \times B(r_0, 4Cc) \to \mathbb{C}$ given as the left hand side of (9). This F is well-defined since f is defined on $B(0, 10Cc) \subset \Phi_0(U_j)$. To guarantee the existence of $r_k = r_k(y)$ for all $y \in B_{g_k}(x,c)$ we have to check that

$$\left| \frac{\nabla_y F}{\partial F / \partial r_k} \right| = \left| \frac{\langle df, \nabla f_k \rangle}{\langle df, v \rangle} \right| \leq 4C,$$

which holds thanks to (8). This gives the existence of r_k in the whole of the ball $B_{g_k}(x,c)$ as well as the bound $|\nabla r_k| \leq 4C$, and hence $|r_k| \leq 8Cc$.

Now our task will be to prove that r_k is asymptotically holomorphic, so we change a geometrical transversality problem into a local one. For this let us compute $\bar\partial r_k$. Recall that f_k is asymptotically holomorphic and f is holomorphic. Differentiate the equality $f(f_k(y)+r_k(y)v)=0$ to get

$$\begin{aligned} 0 &= \bar\partial(f(f_k(y)+r_k(y)v)) = \partial f(z)\cdot(\bar\partial f_k(y)+\bar\partial r_k(y)v) = \\ &= O(k^{-1/2}) + \langle df(z), v\rangle \bar\partial r_k(y), \end{aligned} \tag{10}$$

with $z=f_k(y)+r_k(y)v$. Using (8) we get that $\bar\partial r_k = O(k^{-1/2})$. We already know that $|\nabla r_k| = O(1)$. Differentiating (10) one easily obtains also that $|\nabla\bar\partial r_k| = O(k^{-1/2})$. So r_k is asymptotically holomorphic. We shall achieve transversality for the function

$$h_k = r_k \frac{s_k^0}{s_{k,x}^{\mathrm{ref}}} : B_{g_k}(x,c) \to \mathbb{C},$$

which is also asymptotically holomorphic.

Dividing h_k by an appropriate constant, using the chart Φ_k defined in Lemma 2.6 and scaling the coordinates by a universal constant, we obtain a function $\tilde h_k$ defined on B^+ satisfying the hypothesis of Proposition 2.10, for k large enough. So going back to h_k through universal constants, we find $|w_k|<\delta$ such that $h_k - w_k$ is η-transverse to 0 with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$.

Now we have a direction v and a modulus w_k for a perturbation. The perturbation we give is

$$\tau_{k,x} = (0, -w_k v_1 s_{k,x}^{\mathrm{ref}}, \ldots, -w_k v_{2n+1} s_{k,x}^{\mathrm{ref}}).$$

Let us look at the perturbed map $\hat\phi_k = \mathbb{P}(s_k+\tau_{k,x})$. It is asymptotically holomorphic and γ'-projectizable and γ'-generic of order n, for some $\gamma'>0$, with $|\tau_{k,x}|<c''\delta$ (for $\delta>0$ small enough). Let us check that $\hat\phi_k$ is η-transverse to N with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ and c' a constant depending only on c and the asymptotically holomorphic bounds of s_k. With this, applying Proposition 2.8, the proof in this case is concluded. Only a little problem may appear, that the deformed embedding can become an immersion, but then an arbitrarily small perturbation solves the problem.

The h_k associated to $\hat\phi_k$ is $\hat h_k = h_k - w_k$. The final point is to set up the relationship between the transversality of $\hat h_k$ to 0 and the transversality of $\hat\phi_k$ to N. Note that we have $\hat r_k = \hat h_k \frac{s_{k,x}^{\mathrm{ref}}}{s_k^0}$, $\hat f_k = \Phi_0\circ\hat\phi_k$ and $\hat\pi_k = \hat f_k + \hat r_k v = \pi_k$.

Using that $|s_{k,x}^{\mathrm{ref}}/s_k^0|$ is bounded above and below uniformly and that $|\nabla(s_{k,x}^{\mathrm{ref}}/s_k^0)| = O(1)$, it is easy to prove that if $\hat h_k$ is η-transverse to 0 then $\hat r_k$ is $c_0\eta$-transverse to 0, for some universal constant c_0.

Let $y\in B_{g_k}(x,c)$. If $|\hat r_k(y)|\geq c_0\eta$ then $d(\hat\phi_k(y),N)\geq c_1\eta$, for some universal constant c_1. Otherwise $|\nabla\hat r_k(y)|>c_0\eta$. We shall use Lemma 3.8 for the subspaces $U=(d\hat f_k)_* T_yM$ and $V=T_{\pi_k(y)}Z$ of $\mathbb{C}^{2n+1}$. Let $V'=[v]$. The projections from U to the summands of the decomposition $\mathbb{C}^{2n+1}=V\oplus V'$ are given respectively by $g = d\pi_k\circ(d\hat f_k)^{-1}$ and $h=-v\,d\hat r_k\circ(d\hat f_k)^{-1}$. This follows from $d\pi_k = d\hat f_k + d\hat r_k\, v$ which gives $\mathrm{Id} = d\pi_k\circ(d\hat f_k)^{-1} - v\,d\hat r_k\circ(d\hat f_k)^{-1}$. The map h has a right inverse of norm bounded by $C'\eta^{-1}$, for some universal constant C' (here we use that ϕ_k is generic of order n and that the perturbations are small). It is easy to check that Lemma 3.8 is still valid when V and V' are almost orthogonal (and not just orthogonal), so we have

$$\angle_m((d\hat f_k)_* T_yM, T_{\pi_k(y)}Z) \geq c_2\eta.$$

Push forward the distribution D_N through the chart Φ_0 to a distribution D_Z in $B(0,15Cc)$. Then there exists a constant C'' independent of k such that

$$\angle_M(T_zZ, D_Z(z+\lambda v)) < C'' d(z+\lambda v, Z),$$

for $z \in Z$, $\lambda \in \mathbb{C}$ with $|z| < 14Cc$, $|\lambda| < Cc$. Now use Proposition 3.5 to get

$$\angle_m((d\hat{f}_k)_* T_y M, D_Z(\hat{f}_k(y))) > c_2\eta - C'' d(\hat{f}_k(y), Z).$$

For $d(\hat{f}_k(y), Z) < c_2\eta/2C''$ we get $\angle_m((d\hat{f}_k)_* T_y M, D_Z(\hat{f}_k(y))) > c_2\eta/2$. Passing to the manifold we get $\angle_m((d\hat{\phi}_k)_* T_y M, D_N(\hat{\phi}_k(y)) > c_2'\eta$, whenever $d(\hat{\phi}_k(y), N) < c_1'\eta$, for some universal constants c_1' and c_2'.

To achieve the solution when the codimension of N is $r > 1$, we follow the same ideas than in the precedent case. In this case $f : B(0, 15Cc) \to \mathbb{C}^r$ and one chooses the point z_0 giving the minimum distance from 0 to Z which yields a vector v_1 orthogonal to Z at z_0. Then one completes to an unitary basis $(v_1, \ldots, v_r)$ for the orthogonal to $T_{z_0}Z$. The function $r_k : B_{g_k}(x, c) \to \mathbb{C}^r$ is defined by the condition $f(f_k + r_k^1 v_1 + \ldots + r_k^r v_r) = 0$. The perturbation will be of the form

$$\tau_{k,x} = -(0, w_k^1 v_1^1 s_{k,x}^{\mathrm{ref}} + \cdots + w_k^r v_r^1 s_{k,x}^{\mathrm{ref}}, \ldots, w_k^1 v_1^{2n+1} s_{k,x}^{\mathrm{ref}} + \cdots + w_k^r v_r^{2n+1} s_{k,x}^{\mathrm{ref}}),$$

where $v_i = (v_i^1, \ldots, v_i^{2n+1})$, $i = 1, \ldots, r$ and $w_k = (w_k^1, \ldots, w_k^r) \in \mathbb{C}^r$. The proof above works out in this case. $\square$

4. Asymptotically holomorphic embeddings to grassmannians

Let (M, ω) be a symplectic manifold of integer class and let L stand for the hermitian line bundle with a connection ∇ with curvature $-i\omega$. Let E be a rank r hermitian bundle over M endowed with an hermitian connection. Fix a compatible almost complex structure J on M. In this Section we shall deal with the issue of constructing sequences of embeddings of M into the grassmannian $\mathrm{Gr}(r, N)$ which are asymptotically J-holomorphic in the sense of Definition 1.1. More specifically, we aim to prove the following result from which Theorem 1.4 follows.

Theorem 4.1. *Suppose $N > n + r - 1$ and $r(N - r) > 2n$. Given an asymptotically J-holomorphic sequence of sections s_k of the vector bundles $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ and $\alpha > 0$ then there exists another sequence σ_k verifying that:*

1. $|s_k - \sigma_k|_{C^1, g_k} < \alpha$.
2. $\phi_k = \mathrm{Gr}(\sigma_k)$ *is an asymptotically holomorphic sequence of embeddings in* $\mathrm{Gr}(r, N)$ *for k large enough.*
3. $\phi_k^* \mathcal{U} = E \otimes L^{\otimes k}$*, where $\mathcal{U} \to \mathrm{Gr}(r, N)$ is the universal rank r bundle over the grassmannian.*

Moreover given two asymptotically holomorphic sequences ϕ_k^0 and ϕ_k^1 of embeddings in $\mathrm{Gr}(r, N)$ with respect to two compatible almost complex structures, then for k large enough there exists an isotopy of asymptotically holomorphic embeddings ϕ_k^t connecting ϕ_k^0 and ϕ_k^1.

4.1. **Proof of main result.**

First let us fix some notation. A point $s \in \mathrm{Gr}(r, N)$ corresponds to an r-dimensional subspace $V_s \subset \mathbb{C}^N$. Choosing a basis $s_1, \ldots, s_r$ for V_s, we denote

$$s = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1N} \\ \vdots & & \ddots & \vdots \\ s_{r1} & s_{r2} & \cdots & s_{rN} \end{bmatrix}.$$

This identifies s as the equivalence class of $r \times N$ matrices of rank r under the action of $\mathrm{GL}(r, \mathbb{C})$ on the left. The standard metric g_{Gr} for $\mathrm{Gr}(r, N)$ is the metric induced

by the Fubini-Study metric g_{FS} under the Plücker embedding [GH78, Chapter 1, Section 5]

$$\begin{array}{ccc} \mathrm{Gr}(r,N) & \longrightarrow & \mathbb{P}(\bigwedge^r \mathbb{C}^N) \\ \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} & \mapsto & s_1 \wedge \cdots \wedge s_r. \end{array}$$

We proceed by steps to obtain asymptotically holomorphic embeddings.

Definition 4.2. *Let $\gamma > 0$ and $0 \leq l \leq r$. A sequence of asymptotically J-holomorphic sections $s_k = (s_k^1, \dots, s_k^N)$ of the vector bundles $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ is said to be γ-grassmannizable of order l if for all $x \in M$, $|\bigwedge^l s_k(x)| > \gamma$. It is γ-grassmannizable when it is γ-grassmannizable of order r. (Here $s_k = (s_k^1, \dots, s_k^N)$ is interpreted as a morphism of bundles $\underline{\mathbb{C}}^N \to E \otimes L^{\otimes k}$ and $\bigwedge^l s_k$ is the corresponding l-fold wedge product.)*

If we have the condition of γ-grassmannizability for a section s_k then we obtain a morphism $\phi_k = \mathrm{Gr}(s_k) : M \to \mathrm{Gr}(r, N)$, called the grassmannization of s_k, as follows. At a point x take a basis $(e_1, \dots, e_r)$ for the fibre of E at x. Then

$$\phi_k(x) = \left[s_k^1(x), \dots, s_k^N(x)\right] = \begin{bmatrix} s_k^{11} & s_k^{12} & \cdots & s_k^{1N} \\ \vdots & & \ddots & \vdots \\ s_k^{r1} & s_k^{r2} & \cdots & s_k^{rN} \end{bmatrix}.$$

where $s_k^i(x) = s_k^{1i} e_1 + \cdots + s_k^{ri} e_r$. This is well-defined and independent of the chosen basis.

Definition 4.3. *Let $\eta > 0$ and $0 \leq l \leq n$. A sequence of asymptotically J-holomorphic γ-grassmannizable sections s_k of vector bundles $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ is η-generic of order l, with $\eta > 0$, if given $\mathrm{Gr}(s_k)$ then for all $x \in M$, $|\bigwedge^l \partial \mathrm{Gr}(s_k)(x)|_{g_k} > \eta$.*

In order to prove Theorem 4.1 we shall use the following auxiliar Proposition that will be proved in the following Subsections. Also we state the analogue of Lemma 2.15 which will be proved in Subsection 4.4.

Proposition 4.4. *Suppose $N > n + r - 1$ and $r(N - r) > 2n$. Let s_k be an asymptotically J-holomorphic sequence of sections of the vector bundles $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ and $\alpha > 0$. Then there exists another sequence σ_k verifying:*

1. *$|s_k - \sigma_k|_{C^1, g_k} < \alpha$.*
2. *σ_k is γ-grassmannizable and γ-generic of order n for some $\gamma > 0$.*

Moreover, the result holds for one-parameter families of sections where the sections and the compatible almost complex structures depend continuously on $t \in [0, 1]$.

Lemma 4.5. *Let $\phi_k : M \to \mathrm{Gr}(r, N)$ be a sequence of asymptotically holomorphic embeddings with $\phi_k^* \mathcal{U} = E \otimes L^{\otimes k}$. Then there exists a sequence of asymptotically holomorphic sections s_k of $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$, for k large enough, which is γ-grassmannizable and γ-generic of order n, for some $\gamma > 0$, such that $\phi_k = \mathrm{Gr}(s_k)$. The same holds for continuous one-parameter families of embeddings and compatible almost complex structures.*

Proof of Theorem 4.1. Note that the last property is obvious by the construction. Let us begin with an asymptotically J-holomorphic sequence σ_k of sections of the bundles $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ and perturb it using Proposition 4.4 to obtain an asymptotically holomorphic γ-grassmannizable and γ-generic of order

n sequence of sections s_k. The first property implies that $\phi_k = \mathrm{Gr}(s_k)$ is well-defined, the second that it is an immersion. To get an embedding we use that 2dim $M <$ dim $\mathrm{Gr}(r,N) = 2r(N-r)$ to find a generic C^p-perturbation of norm less than $O(k^{-1/2})$ to get rid of the self-intersections and keeping the asymptotic holomorphicity, the grassmannizability and the genericity of order n. Now we only have to check that the sequence $\phi_k = \mathrm{Gr}(s_k)$ verifies the required conditions in Definition 1.1.

Choose a point $x \in M$ and trivialize E in a neighborhood of x by fixing an orthonormal basis $e_1, \dots, e_r$. Now by a rotation with an element of $U(N)$ acting on $\mathbb{C}^N$ and an element of $U(r)$ acting on E, we can assure that

$$s_k(x) = \begin{pmatrix} s_k^{11}(x) & 0 & \dots & & & \dots & 0 \\ 0 & s_k^{22}(x) & 0 & \dots & & & 0 \\ & & \ddots & & & & \\ 0 & \dots & & 0 & \dots & & 0 \\ 0 & \dots & & s_k^{rr}(x) & 0 & \dots & 0 \end{pmatrix} \tag{11}$$

where s_k^{ij} are sections of $L^{\otimes k}$. This corresponds to an isometric transformation of $\mathrm{Gr}(r,N)$. The γ-grassmannizable property implies that $|s_k^{11} \cdots s_k^{rr}| \geq \gamma$. By the asymptotic holomorphicity bounds it is $|s_k| = O(1)$, so that $|s_k^{ii}| \geq \gamma/C$, for some universal constant C. Therefore on a ball $B_{g_k}(x,c)$ of fixed universal radius c, the first $r \times r$ minor of $s_k(y)$ has an inverse of norm bounded by $C'\gamma^{-1}$, for some universal constant C'.

Let $v_1, \dots, v_N$ be the canonical basis of $\mathbb{C}^N$. As $\phi_k(x) = \Pi_0 = \begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix}$, we consider the standard local chart for $\mathrm{Gr}(r,N)$ around Π_0 for the open set $U_0 = \{\Pi \big| \Pi \cap [v_{r+1}, \dots, v_N] = \{0\}\}$, given by

$$\begin{aligned} \Phi_0 : U_0 \quad &\rightarrow \quad \mathbb{C}^{r\times(N-r)} \\ \begin{bmatrix} s_{11} & \cdots & s_{1N} \\ \vdots & \ddots & \vdots \\ s_{r1} & \cdots & s_{rN} \end{bmatrix} \quad &\mapsto \quad \begin{pmatrix} s_{11} & \dots & s_{1r} \\ \vdots & \ddots & \vdots \\ s_{r1} & \dots & s_{rr} \end{pmatrix}^{-1} \begin{pmatrix} s_{1,r+1} & s_{1,r+2} & \dots & s_{1N} \\ \vdots & & \ddots & \vdots \\ s_{r,r+1} & s_{r,r+2} & \dots & s_{rN} \end{pmatrix} \end{aligned}$$

It is easy to check that Φ_0 is an isometry at the point Π_0.

The application $f_k = \Phi_0 \circ \phi_k$ is given by

$$\begin{aligned} f_k : B_{g_k}(x,c) \quad &\rightarrow \quad \mathbb{C}^{r\times(N-r)} \\ y \quad &\mapsto \quad \begin{pmatrix} s_k^{11}(y) & \dots & s_k^{1r}(y) \\ \vdots & \ddots & \vdots \\ s_k^{r1}(y) & \dots & s_k^{rr}(y) \end{pmatrix}^{-1} \begin{pmatrix} s_k^{1,r+1}(y) & \dots & s_k^{1N}(y) \\ \vdots & \ddots & \vdots \\ s_k^{r,r+1}(y) & \dots & s_k^{rN}(y) \end{pmatrix} \end{aligned}$$

We can compute the bounds required in Definition 1.1 using f_k instead of ϕ_k. Now the arguments in the proof of Theorem 2.11 carry over verbatim. For the isotopy result we use Lemma 4.5. □

4.2. **Construction of γ-grassmannizable sections.** Our objective is to prove the following perturbation result:

Proposition 4.6. *Suppose $N > n+r-1$. Let s_k be an asymptotically J-holomorphic sequence of sections of the vector bundles $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ which is γ-grassmannizable of order l, for some $\gamma > 0$. Then given $\alpha > 0$, there exists an asymptotically J-holomorphic sequence of sections σ_k verifying:*

1. $|s_k - \sigma_k|_{C^1, g_k} < \alpha$.
2. *σ_k is η-grassmannizable of order $l+1$ for some $\eta > 0$.*

Moreover, the result can be extended to continuous one-parameter families depending continuously of $t \in [0,1]$.

Proof. Again we use the globalization argument described in Proposition 2.8. Let us do the non-parametric case, the other one being a trivial extension by now. Define the local and C^0-open property $\mathscr{P}(\epsilon, x)$ as $|\bigwedge^{l+1} s_k(x)| > \epsilon$. We only need to find for a point $x \in M$ a section $\tau_{k,x}$ with Gaussian decay away from x, assuring that $s_k + \tau_{k,x}$ verifies $\mathscr{P}(\eta, y)$ in a ball of universal g_k-radius c.

Choose a point $x \in M$. Fix an orthonormal basis $e_1, \dots, e_r$ trivializing E in a neighbourhood of x, so s_k may be interpreted as a morphism $\mathbb{C}^N \to \mathbb{C}^r \otimes L^{\otimes k}$. By a rotation with an element of $U(N)$ on $\mathbb{C}^N$ and an element of $U(r)$ on E, we can assure that

$$s_k(x) = \begin{pmatrix} s_k^{11}(x) & 0 & \dots & & & \dots & 0 \\ 0 & s_k^{22}(x) & 0 & \dots & & & 0 \\ & & & \ddots & & & \\ 0 & \dots & & & 0 & \dots & 0 \\ 0 & \dots & & s_k^{rr}(x) & 0 & \dots & 0 \end{pmatrix}$$

with $|s_k^{11}(x) \cdots s_k^{ll}(x)| \geq \gamma$. So $|s_k^{11} \cdots s_k^{ll}| > \gamma/2$ on a ball $B_{g_k}(x,c)$ of fixed radius c. Let $s_{k,x}^{\mathrm{ref}}$ be the sections given by Lemma 2.5 and define $\theta_k = s_k^{11} \cdots s_k^{ll} s_{k,x}^{\mathrm{ref}}$. Clearly $|\theta_k| > c_s\gamma/2$ on $B_{g_k}(x,c)$. Consider the family of functions

$$M_k^p = s_k^{11} \cdots s_k^{ll} s_k^{l+1,p}, \quad l+1 \leq p \leq N.$$

These are components of $\bigwedge^{l+1} s_k$. If we perturb s_k so that the norm of $M_k = (M_k^{l+1}, \dots, M_k^N)$ is bigger than $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ then we have finished. For this we define $g_k = (g_k^{l+1}, \dots, g_k^N) = \left(\frac{M_k^{l+1}}{\theta_k}, \dots, \frac{M_k^N}{\theta_k}\right) = \left(\frac{s_k^{l+1,l+1}}{s_{k,x}^{\mathrm{ref}}}, \dots, \frac{s_k^{l+1,N}}{s_{k,x}^{\mathrm{ref}}}\right)$. We obtain, scaling the coordinates by universal constants if necessary, $g_k : B^+ \to \mathbb{C}^{N-l}$ which is asymptotically holomorphic. As $n < N-l$, we can find $|w_k| < \delta$ such that $|g_k - w_k| > \delta(\log(\delta^{-1}))^{-p}$. Then we obtain that $|(M_k^{l+1} - w_k^{l+1}\theta_k, \dots, M_k^N - w_k^N\theta_k)| > \eta = c'\delta(\log(\delta^{-1}))^{-p}$, for some universal c'. This perturbation term is achieved by adding the section $\tau_{k,x} = -(0, \overset{(l)}{\dots}, 0, w_k^{l+1} e_{l+1} s_{k,x}^{\mathrm{ref}}, \dots, w_k^N e_{l+1} s_{k,x}^{\mathrm{ref}})$ of the bundles $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$. This finishes the proof. □

Remark 4.7. *We cannot improve the condition* $N > n + r - 1$ *in Proposition 4.6. As we shall see in Section 5, we expect the locus of points of* M *where the rank of* $s_k : \underline{\mathbb{C}}^N \to E \otimes L^{\otimes k}$ *is not maximum to have codimension* $N - r + 1$.

4.3. **Inductive construction of sections γ-generic of order l.** Now we study the problem of perturbing the sequence s_k to achieve genericity of order n. The result to be proved is the following.

Proposition 4.8. *Suppose* $r(N-r) > 2n$. *Let* s_k *be an asymptotically* J*-holomorphic sequence of sections of the vector bundles* $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$, *which is* γ*-grassmannizable and* γ*-generic of order* l. *Then given* $\alpha > 0$, *there exists an asymptotically* J*-holomorphic sequence of sections* σ_k *verifying:*

1. $|s_k - \sigma_k|_{C^1, g_k} < \alpha$.
2. σ_k *is* η*-generic of order* $l+1$ *for some* $\eta > 0$.

Moreover, this result can be extended to continuous one-parameter families of sections and almost complex structures.

Proof. Define the property $\mathscr{P}(\epsilon, x)$ for a section s_k which is $\gamma/2$-grassmannizable and $\gamma/2$-generic of order l as $|\bigwedge^{l+1} \partial \mathrm{Gr}(s_k)(x)| > \epsilon$. A perturbation of our initial section verifies the hypothesis if we perturb by adding sections of C^1 norm smaller than $\gamma/2C$, C some universal constant. For applying Proposition 2.8 we need to

build, for $0<\delta<\gamma/2Cc''$, a local perturbation $\tau_{k,x}$ with $|\tau_{k,x}|<c''\delta$ and Gaussian decay with the property $\mathscr{P}(\eta,y)$ on $B_{g_k}(x,c)$ with $\eta=c'\delta(\log(\delta^{-1}))^{-p}$.

Choose a point $x\in M$. By a rotation with an element of $U(N)$ acting on $\mathbb{C}^N$ and an element of $U(r)$ acting on E, we can assure that $s_k(x)$ is as in (11). By the γ-grassmannizability, $|s_k^{11}(x)\cdots s_k^{rr}(x)|\geq\gamma$. The asymptotically holomorphic bounds imply that $|s_k|=O(1)$, so that $|s_k^{ii}(x)|\geq\gamma/C$ for some universal constant C. There is a fixed universal radius c such that the first $r\times r$ minor of $s_k(y)$ has an inverse of norm bounded by $C'\gamma^{-1}$, for some universal constant C', on $B_{g_k}(x,c)$. Then we can use the trivialization Φ_0 to define the applications

$$\begin{array}{rcl} f_k: B_{g_k}(x,c) & \to & \mathbb{C}^{r\times(N-r)} \\ y & \mapsto & \begin{pmatrix} s_k^{11}(y) & \dots & s_k^{1r}(y) \\ \vdots & & \vdots \\ s_k^{r1}(y) & \dots & s_k^{rr}(y) \end{pmatrix}^{-1} \begin{pmatrix} s_k^{1,r+1}(y) & \dots & s_k^{1N}(y) \\ \vdots & & \vdots \\ s_k^{r,r+1}(y) & \dots & s_k^{rN}(y) \end{pmatrix} \end{array}$$

Now consider the sections $s_{k,x}^{\mathrm{ref}}$ of Lemma 2.5. We define the applications

$$\begin{array}{rcl} \tilde{f}_k: B_{g_k}(x,c) & \to & \mathbb{C}^{r\times(N-r)} \\ y & \mapsto & \dfrac{1}{s_{k,x}^{\mathrm{ref}}(y)} \begin{pmatrix} s_k^{1,r+1}(y) & s_k^{1,r+2}(y) & \dots & s_k^{1N}(y) \\ \vdots & & \vdots & \\ s_k^{r,r+1}(y) & s_k^{r,r+2}(y) & \dots & s_k^{rN}(y) \end{pmatrix} \end{array}$$

Clearly $f_k=\Psi\circ\tilde{f}_k$ where $\Psi: B_{g_k}(x,c)\to\mathrm{GL}(r,\mathbb{C})$ satisfies $|\Psi|=O(1)$, $|\Psi^{-1}|=O(1)$, $|\nabla\Psi|=O(1)$ and $|\nabla\Psi^{-1}|=O(1)$. Therefore it is enough to get a perturbation which has $|\bigwedge^{l+1}\partial\tilde{f}_k|>\eta$ on $B_{g_k}(x,c)$.

Spreading out the entries of the matrix $\tilde{f}_k$ in one row we can write $\tilde{f}_k(y)=(\tilde{f}_k^{11}(y),\dots,\tilde{f}_k^{r,N-r}(y))$. Using the local forms $dz_k^1,\dots,dz_k^n$, we may write

$$\partial\tilde{f}_k=(u_k^{111}dz_k^1+u_k^{112}dz_k^2+\cdots+u_k^{11n}dz_k^n,\ \dots,u_k^{r,N-r,1}dz_k^1+\cdots+u_k^{r,N-r,n}dz_k^n),$$

for some u_k^{ijl}. Using a unitary transformation of $U(n)$ on the complex Darboux coordinate chart and relabeling horizontally the coordinates, we can suppose that

$$\partial\tilde{f}_k(x)=\begin{pmatrix} u_k^{11}(x) & * & \dots & & & \dots & * \\ 0 & u_k^{22}(x) & * & \dots & & & * \\ 0 & \dots & \ddots & * & \dots & & * \\ 0 & \dots & 0 & u_k^{nn}(x) & * & \dots & * \end{pmatrix}, \tag{12}$$

where $|u_k^{11}(x)\cdots u_k^{ll}(x)|>\gamma/C_0$, C_0 a universal constant. The relabeling is given by a function $\alpha\in\{1,\dots,r(N-r)\}\mapsto(i(\alpha),j(\alpha))\in\{1,\dots,r\}\times\{1,\dots,N-r\}$. Shrinking c if necessary we can assure that $|(\partial\tilde{f}_k^1\wedge\cdots\wedge\partial\tilde{f}_k^l)_{dz_k^1\wedge\cdots\wedge dz_k^l}|>\gamma/2C_0$ for all the points of the ball $B_{g_k}(x,c)$. Now we construct the $(l+1)$-form

$$\theta_k(y)=(\partial\tilde{f}_k^1\wedge\cdots\wedge\partial\tilde{f}_k^l)_{dz_k^1\wedge\cdots\wedge dz_k^l}\wedge dz_k^{l+1}.$$

and the family of $(l+1)$-forms

$$M_k^p=(\partial\tilde{f}_k^1\wedge\cdots\wedge\partial\tilde{f}_k^l\wedge\partial\tilde{f}_k^p)_{dz_k^1\wedge\cdots\wedge dz_k^l\wedge dz_k^{l+1}},\quad l+1\leq p\leq r(N-r),$$

which are components of $\bigwedge^{l+1}\partial\tilde{f}_k$. If we perturb so that the norm of $M_k=(M_k^{l+1},\dots,M_k^{r(N-r)})$ gets bigger than $\eta=c'\delta(\log(\delta^{-1}))^{-p}$ then we are done. We define the following sequence of asymptotically holomorphic applications: $h_k=\left(\frac{M_k^{l+1}}{\theta_k},\dots,\frac{M_k^{r(N-r)}}{\theta_k}\right)$. So we obtain, scaling the coordinates by universal constants if necessary, $h_k: B^+\to\mathbb{C}^{r(N-r)-l}$ which is asymptotically holomorphic. As $n<r(N-r)-l$ we can find $|w_k|<\delta$ such that $|h_k-w_k|>\delta(\log(\delta^{-1}))^{-p}$. Thus $|(M_k^{l+1}-$

$w_k^{l+1}\theta_k, \dots, M_k^{r(N-r)} - w_k^{r(N-r)}\theta_k)| > \eta = c'\delta(\log(\delta^{-1}))^{-p}$. The perturbation term $-(w_k^{l+1}\theta_k, \dots, w_k^{r(N-r)}\theta_k)$ is achieved by adding the section

$$\tau_{k,x} = -(0, \overset{(r)}{\dots}, 0, \sum_{j(\alpha)=r+1, \alpha>l} w_k^{\alpha} z_{l+1} e_{i(\alpha)} s_{k,x}^{\mathrm{ref}}, \dots, \sum_{j(\alpha)=N, \alpha>l} w_k^{\alpha} z_{l+1} e_{i(\alpha)} s_{k,x}^{\mathrm{ref}}).$$

This finishes the proof in the non-parametric case. The other case is left to the reader. □

4.4. **Lifting asymptotically holomorphic embeddings in grassmannians.**

This Subsection is devoted to a proof of Lemma 4.5, which states that any asymptotically holomorphic embedding into a grassmannian is of the form provided by Theorem 4.1.

Proof of Lemma 4.5. Suppose that we have a sequence of γ-asymptotically holomorphic embeddings $\phi_k : M \to \mathrm{Gr}(r, N)$, for some $\gamma > 0$, with $\phi_k^* \mathcal{U} = E \otimes L^{\otimes k}$, where $\mathcal{U}$ is the universal rank r bundle over the grassmannian. The dual of $\mathcal{U}$ is given by

$$\mathcal{U}^* = \{(\Pi, s) \big| s \in \Pi\} \subset \mathrm{Gr}(r, N) \times \mathbb{C}^N = \underline{\mathbb{C}}^N,$$

interpreted as a sub-bundle of the trivial bundle $\underline{\mathbb{C}}^N$. We consider the sequence of hermitian bundles, $E_k = \phi_k^* \mathcal{U}^* \otimes E \otimes L^{\otimes k} = \mathrm{End}\, E \subset \mathbb{C}^N \otimes E \otimes L^{\otimes k}$. We look for sequences of sections s_k of E_k which are σ-grassmannizable of order n such that they are asymptotically holomorphic when considered as sections of $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$. Let $S_k^l = \mathrm{Tr}(\bigwedge^l s_k)$, which is an asymptotically holomorphic sequence of sections of the trivial vector bundle $\underline{\mathbb{C}}$. We want to prove that $|S_k^r| \geq \sigma$ for k large. We shall prove that we can find sequences s_k with $|S_k^l| \geq \eta_l$, for some $\eta_l > 0$, by induction on l.

Suppose that s_k is an asymptotically holomorphic sequence of sections of E_k such that $|S_k^l| \geq \gamma$. Let $\mathscr{P}(\epsilon, x)$ be the C^1-open property for sequences of sections s_k of E_k given as $S_k^{l+1} = \mathrm{Tr}(\bigwedge^{l+1} s_k)$ is ϵ-transverse to 0 at x.

Let $x \in M$. We want to find a local perturbation with Gaussian decay obtaining the property $\mathscr{P}(\eta, y)$ in a ball of universal g_k-radius c around x. For this, define the local sections σ_k of $\phi_k^* \mathcal{U}^* \otimes E \subset \mathbb{C}^N \otimes E$ as follows. Locally, σ_k is a map

$$\sigma_k : B_{g_k}(x, c) \to \mathbb{C}^N \otimes \mathbb{C}^r,$$

such that for $y \in B_{g_k}(x, c)$, $\sigma_k(y)$ is a $N \times r$ matrix, i.e. a linear map $\sigma_k(y) : \mathbb{C}^N \to \mathbb{C}^r$. The point $\phi_k(y) \in \mathrm{Gr}(r, N)$ corresponds to the image of the embedding $\sigma_k(y)^T : \mathbb{C}^r \to \mathbb{C}^N$. Note that one may identify the tangent space $T_{\phi_k(y)} \mathrm{Gr}(r, N)$ to the set of linear maps $\mathbb{C}^N \to \mathbb{C}^r$ which are zero on $\mathrm{im}(\sigma_k(y)^T)$, i.e. maps φ such that $\varphi\, \sigma_k(y)^* = 0$. With this, $\nabla \sigma_k = \nabla \phi_k + (\nabla \sigma_k \sigma_k^*) \sigma_k$. So it is natural to require $(\nabla_r \sigma_k(y)) \sigma_k(y)^* = 0$, for any $y \in B_{g_k}(x, c)$, where r is the radial vector field from x. We fix $\sigma_k(x)$ satisfying $\sigma_k(x) \sigma_k(x)^* = \boldsymbol{I}$. This determines σ_k uniquely. The following bounds are proved as in Subsection 2.5,

$$\begin{aligned} &\sigma_k(y)\sigma_k(y)^* = \boldsymbol{I}, \ |\sigma_k(y)| = O(1), \ |\nabla \sigma_k(y)| = O(1 + d_k(x, y)), \\ &|\bar{\partial} \sigma_k(y)| = O(k^{-1/2}(1 + d_k(x, y))), \ |\nabla \bar{\partial} \sigma_k(y)| = O(k^{-1/2}(1 + d_k(x, y))). \end{aligned}$$

Trivialize E in a ball $B_{g_k}(x, c)$, so that $s_k / s_{k,x}^{\mathrm{ref}}$ can be considered as an application $B_{g_k}(x, c) \to \mathbb{C}^{r \times N}$. Define the application

$$f_k = \frac{s_k \sigma_k^*}{s_{k,x}^{\mathrm{ref}}} : B_{g_k}(x, c) \to \mathbb{C}^{r \times r},$$

so that $f_k\, \sigma_k = s_k / s_{k,x}^{\mathrm{ref}}$. Then f_k is asymptotically holomorphic and we may check property $\mathscr{P}(\eta, y)$ for f_k instead of s_k. Let $F_i = \mathrm{Tr}(\bigwedge^i f_k)$, so that $|F_l| \geq C\gamma$ for

some universal constant C. For any $w \in \mathbb{C}$ we have

$$\mathrm{Tr}(\bigwedge^{l+1}(f_k - w\boldsymbol{I})) = F_{l+1} - w(n-l)F_l + w^2\binom{n-l+1}{2}F_{l-1} + \ldots + (-w)^{l+1}\binom{n}{l+1}F_0$$

By the standard argument, we may obtain $|w| < \delta$ such that $\frac{F_{l+1}}{F_l} - w$ is η-transverse to 0, with $\eta = \delta(\log(\delta^{-1}))^{-p}$, in $B_{g_k}(x,c)$. Then it is easy to see that $\mathrm{Tr}(\bigwedge^{l+1}(f_k - \frac{w}{n-l}\boldsymbol{I})$ is $c'\eta$-transverse to 0, where c' is a universal constant. The perturbation

$$\tau_{k,x} = -\frac{w}{n-l}\sigma_k s_{k,x}^{\mathrm{ref}}$$

is a sequence of sections of $E_k = \phi_k^*\mathcal{U}^* \otimes E \otimes L^{\otimes k}$, with Gaussian decay such that $|\tau_{k,x}| < c''\delta$ and $s_k + \tau_{k,x}$ satisfies $\mathscr{P}(\eta, y)$ for $y \in B_{g_k}(x,c)$, with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$. By Proposition 2.8, there exists an asymptotically holomorphic sequence of sections of E_k, which we denote by s_k again, such that $S_k^{l+1} = \mathrm{Tr}(\bigwedge^{l+1} s_k)$ is η-transverse to 0, for some $\eta > 0$. For k large enough, the zeroes of S_k^{l+1} is a symplectic submanifold representing the trivial homology class, hence the empty set. So $|S_k^{l+1}| \geq \eta$. This completes the proof. The extension to the one-parameter case is trivial. □

4.5. **Zero sets of vector bundles.** Following the ideas of Subsection 3.3 and using Proposition 3.10 we can prove the following two results

Theorem 4.9. *Given $\phi_k = \mathrm{Gr}(s_k)$, where s_k is a sequence of asymptotically holomorphic sections of $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$, which are γ-grassmannizable and γ-generic of order n, for some $\gamma > 0$. Fix a holomorphic submanifold V in $\mathrm{Gr}(r,N)$. Then for any $\alpha > 0$ there exists a sequence of asymptotically holomorphic sections σ_k of $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ such that*

1. $|\sigma_k - s_k|_{g_k, C^1} < \alpha$.
2. *$\mathrm{Gr}(\sigma_k)$ is an η-asymptotically holomorphic embedding in $\mathrm{Gr}(r,N)$ which is ϵ-transverse to V, with $\eta > 0$ and $\epsilon > 0$ independent of k. In the case $\dim M + \dim V < 2r(N-r)$ we have that $d_{Gr}(\phi_k(M), V)) > \epsilon$, for k large enough.*

Moreover the result can be extended to one-parameter continuous families of complex submanifolds $(V_t)_{t\in[0,1]}$, taking in this case as starting point a continuous family $\phi_{t,k} = \mathrm{Gr}(s_{t,k})$, where $s_{t,k}$ is a continuous family of γ-grassmannizable and γ-generic of order n sequences asymptotically J_t-holomorphic.

Proof. The proof is similar to that of Theorem 3.12. We just briefly point out the differences. For simplicity we suppose that the codimension of V is 1.

For $x \in M$, we may suppose that $s_k(x)$ is as in (11). We use the chart Φ_0 to get the local maps

$$\begin{aligned} f_k = \Phi_0 \circ \phi_k : B_{g_k}(x,c) &\rightarrow \mathbb{C}^{r\times(N-r)} \\ y &\mapsto \begin{pmatrix} s_k^{11}(y) & \ldots & s_k^{1r}(y) \\ \vdots & & \vdots \\ s_k^{r1}(y) & \ldots & s_k^{rr}(y) \end{pmatrix}^{-1} \begin{pmatrix} s_k^{1,r+1}(y) & \ldots & s_k^{1N}(y) \\ \vdots & & \vdots \\ s_k^{r,r+1}(y) & \ldots & s_k^{rN}(y) \end{pmatrix} \end{aligned}$$

This time we have a vector $v \in \mathbb{C}^{r\times(N-r)}$. We define the functions $h_k : B_{g_k}(x,c) \rightarrow \mathbb{C}$ by the condition

$$f\left(f_k + r_k s_{k,x}^{\mathrm{ref}} \begin{pmatrix} s_k^{11} & \cdots & s_k^{1r} \\ \vdots & \ddots & \vdots \\ s_k^{r1} & \cdots & s_k^{rr} \end{pmatrix}^{-1} \begin{pmatrix} s_k^{11}(x) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_k^{rr}(x) \end{pmatrix} \begin{pmatrix} v^{11} & \cdots & v^{1,N-r} \\ \vdots & \ddots & \vdots \\ v^{r1} & \cdots & v^{r,N-r} \end{pmatrix} \right) = 0,$$

and prove that they are asymptotically holomorphic. Then we find $|w_k| < \delta$ such that $h_k - w_k$ is η-transverse to 0 with $\eta = c'\delta(\log(\delta^{-1}))^{-p}$. Finally the perturbation will be

$$\tau_{k,x} = -\begin{pmatrix} 0 & \cdots & 0 & w_k v^{11} s_{k,x}^{\mathrm{ref}} & \cdots & w_k v^{1,N-r} s_{k,x}^{\mathrm{ref}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_k v^{r1} s_{k,x}^{\mathrm{ref}} & \cdots & w_k v^{r,N-r} s_{k,x}^{\mathrm{ref}} \end{pmatrix}.$$

The arguments run parallel to those in the proof of Theorem 3.12, although the constants have to be arranged suitably, but we leave this task to the careful reader. □

We call universal planes to the zero sets of algebraic sections transverse to zero of the universal bundle $\mathcal{U}$ over the grassmannian $\mathrm{Gr}(r, N)$. Now we can deduce the main result of [Au97].

Corollary 4.10. *Let (M, ω) be a compact symplectic manifold of integer class. Let E be a hermitian vector bundle over M. Then for k large enough there exist symplectic submanifolds obtained as zero sets of the bundles $E \otimes L^{\otimes k}$. Moreover, perhaps by increasing k, we can assure that all the symplectic submanifolds constructed as transverse intersections of asymptotically holomorphic sequences with a fixed universal plane are isotopic. The isotopy can be made through symplectomorphisms.*

The proof follows the steps of the proof of Corollary 3.13. Remark also that the result is a corollary of Theorem 5.4 to be proved in Section 5.

5. Determinantal submanifolds of closed symplectic manifolds

Let (M, ω) be a symplectic 4-manifold of integer class, endowed with a compatible almost complex structure. Let E and F be two vector bundles of ranks r_e and r_f, respectively. Recall that for any morphism $\varphi : E \to F$ we have defined in Definition 1.5 the r-determinantal set as

$$\Sigma^r(\varphi) = \{x \in M \big| \operatorname{rank} \varphi_x = r\}.$$

We want to prove Theorem 1.6, which allows to construct $\Sigma^r(\varphi)$ as a symplectic submanifold, after twisting E and F with large powers of L. The solution to this problem goes through embedding M in a product of two grassmannians and cutting its image with suitable "generalized Schur cycles". We shall do this in next Section.

Remark 5.1. *A direct approach to proving Theorem 1.6 consists on reducing it to Auroux' case by taking the r-fold wedge product of φ_k,*

$$\begin{aligned} \bigwedge^r \varphi_k : \bigwedge^r E \otimes (L^*)^{\otimes k} &\to \bigwedge^r F \otimes L^{\otimes k} \\ s_1 \wedge \cdots \wedge s_r &\mapsto \varphi_k(s_1) \wedge \cdots \wedge \varphi_k(s_r). \end{aligned}$$

So the zero set of $\bigwedge^r \varphi_k$ is generically a stratified submanifold $\Sigma^0(\varphi_k) \bigcup \ldots \bigcup \Sigma^r(\varphi_k)$. If we suppose that φ_k is an asymptotically J-holomorphic sequence of sections of the bundle $E^* \otimes F \otimes L^{\otimes 2k}$, one could try to use Donaldson's techniques to obtain a new sequence of sections transverse in an adequate sense to assure the symplecticity. The following example shows the main obstacle to this approach. Take a symplectic 4-manifold in the hypothesis of Theorem 1.6 with two hermitian vector bundles E and F of rank 2. Using Auroux' techniques we can assure that the zero sets of φ_k are η-transverse to 0, for some $\eta > 0$. When we go to $\bigwedge^2 \varphi_k$, the condition to be satisfied is

$$|\bar\partial \bigwedge^2 \varphi_k| < |\partial \bigwedge^2 \varphi_k|.$$

However, at any $x \in Z(\varphi_k)$ we obtain $|\bar\partial \bigwedge^2 \varphi_k(x)| = |\partial \bigwedge^2 \varphi_k(x)| = 0$, so we cannot impose a global transversality property for the section $\bigwedge^2 \varphi_k$. This case is

very similar to that in [Do99] *and can be treated with an "ad hoc" argument, but more general cases do not admit a treatment based on the use of normal forms of the singularities, because for higher dimensions the problem becomes intractable* [Ar82].

5.1. **Bigrassmannian embeddings.** The idea to prove Theorem 1.6 is based in the following observations. Choose two sequences of sections s_k^e and s_k^f of the bundles $\mathbb{C}^N \otimes E^* \otimes L^{\otimes k}$ and $\mathbb{C}^N \otimes F \otimes L^{\otimes k}$ respectively, which are γ-grassmannizable and γ-generic of order n, for some $\gamma > 0$, providing by Theorem 4.1, asymptotically holomorphic sequences of embeddings $\mathrm{Gr}(s_k^e)$ and $\mathrm{Gr}(s_k^f)$ of M in $\mathrm{Gr}(r_e, N)$ and $\mathrm{Gr}(r_f, N)$, respectively, for N a large integer number.

Performing the cartesian product we obtain an asymptotically holomorphic sequence of embeddings of M into the bigrassmannian $\mathrm{Bi}(r_e, r_f, N) = \mathrm{Gr}(r_e, N) \times \mathrm{Gr}(r_f, N)$,

$$\phi_k = \mathrm{Gr}(s_k^e) \times \mathrm{Gr}(s_k^f) : M \to \mathrm{Gr}(r_e, N) \times \mathrm{Gr}(r_f, N) = \mathrm{Bi}(r_e, r_f, N).$$

Let $\mathcal{U}_e$ and $\mathcal{U}_f$ be the universal bundles over $\mathrm{Gr}(r_e, N)$ and $\mathrm{Gr}(r_f, N)$ respectively, which are very ample. Define $\pi_e : \mathrm{Bi}(r_e, r_f, N) \to \mathrm{Gr}(r_e, N)$ as the projection onto the first factor (and analogously π_f). Therefore $\mathcal{U}_{ef} = \pi_e^*(\mathcal{U}_e) \otimes \pi_f^*(\mathcal{U}_f) = \mathcal{U}_{ef}$ is very ample on $\mathrm{Bi}(r_e, r_f, N)$. Recall that $\mathrm{Gr}(s_k^e)^*(\mathcal{U}_e) = E^* \otimes L^{\otimes k}$ and $\mathrm{Gr}(s_k^f)^*(\mathcal{U}_f) = F \otimes L^{\otimes k}$. Then $\phi_k^* \mathcal{U}_{ef} = E^* \otimes F \otimes L^{\otimes 2k}$. $\mathcal{U}_{ef}$ has a holomorphic section s verifying that:

1. $D_r = \Sigma^r(s)$ is an open complex submanifold in $\mathrm{Bi}(r_e, r_f, N)$.
2. $\operatorname{codim}_{\mathbb{C}} D_r = (r_e - r)(r_f - r)$.

If we assure that, for each r, ϕ_k is transverse to D_r with an angle $\epsilon > 0$ independent of k, we have finished the proof of Theorem 1.6 by Proposition 3.10. This is carried out as follows.

Lemma 5.2. *Let $\phi_k : M \to \mathrm{Bi}(r_e, r_f, N)$ be a γ-asymptotically holomorphic sequence of embeddings. Suppose that ϕ_k is σ-transverse to D_r. Then there exists $\epsilon > 0$, depending only on γ, σ and the universal bounds of the derivatives of the sequence, such that ϕ_k is $\sigma/2$-transverse to $D_{r'}$, $r' > r$, when we restrict to an ϵ-neighborhood of D_r.*

In other words we do not have to care about the behaviour of the angle near the border of the strata.

Proof. Choose a point $x \in D_r \bigcap \phi_k(M)$. Recall that by σ-transversality, the minimum angle between $T_x D_r$ and $T_x \phi_k(M)$ is greater than σ. We trivialize $\mathrm{Bi}(r_e, r_f, N)$ by a chart Φ_0 defined as the cartesian product of two standard charts in the grassmannians, which is an isometry at the origin and verifies that $\Phi_0(x) = 0$, namely,

$$\Phi_0 : \mathrm{Bi}(r_e, r_f, N) \to \mathbb{C}^{r_e(N - r_e)} \times \mathbb{C}^{r_f(N - r_f)}.$$

Since D_r is contained in the closure of $D_{r'}$, we have

$$|y| < \delta \Rightarrow \angle_M(T_0\Phi_0(D_r), T_y\Phi_0(D_{r'})) < c_D\delta, \ \forall y \in B(0, c_u) \bigcap \Phi_0(D_{r'}). \tag{13}$$

The angles are measured with respect to the standard euclidean metric which is close to that induced by the bigrassmannian if we choose c_u small enough. Here c_D is universal. Also by the asymptotic holomorphicity bounds of ϕ_k we know that

$$\begin{aligned} &|y| < \delta \Rightarrow \angle_M(T_0\Phi_0(\phi_k(M)), T_y\Phi_0(\phi_k(M))) < c_\phi\delta, \\ &\forall y \in B(0, c_u) \bigcap \Phi_0(\phi_k(M)), \end{aligned} \tag{14}$$

where c_ϕ is universal. Now Proposition 3.5 says that

$$\angle_m(T_0\Phi_0(D_r), T_0\Phi_0(\phi_k(M))) \le \angle_M(T_0\Phi_0(D_r), T_y\Phi_0(D_{r'})) +$$

$$+\angle_m(T_y\Phi_0(D_{r'}), T_y\Phi_0(\phi_k(M))) + \angle_M(T_y\Phi_0(\phi_k(M)), T_0\Phi_0(\phi_k(M))).$$

Using inequalities (13) and (14) and remembering that all the angles have to be measured with respect to the bigrassmannian metric (which is related to the standard metric in the ball $B(0, c_u)$ by non zero universal constants), we get the required result. □

With Lemma 5.2 the proof of Theorem 1.6 reduces to the following result, whose proof is similar to that of Theorem 4.9.

Proposition 5.3. *Let s_k^e and s_k^f be two asymptotically holomorphic sequences of the vector bundles $E^*\otimes L^{\otimes k}$ and $F\otimes L^{\otimes k}$ which are γ-grassmannizable and γ-generic of order n, defining so an asymptotically holomorphic embedding in $\mathrm{Bi}(r_e, r_f, N)$. Fix an algebraic open submanifold V in $\mathrm{Bi}(r_e, r_f, N)$ with compactification $\bar{V} = V\bigcup W$. Then for any $\epsilon, \alpha > 0$, there exist $\eta > 0$ and two asymptotically holomorphic sequences σ_k^e and σ_k^f of sections of the vector bundles $E^*\otimes L^{\otimes k}$ and $F\otimes L^{\otimes k}$ respectively, verifying:*

1. *$|\sigma_k^e - s_k^e|_{g_k, C^1} < \alpha$ and $|\sigma_k^f - s_k^f|_{g_k, C^1} < \alpha$.*
2. *$\phi_k = \mathrm{Gr}(\sigma_k^e)\times \mathrm{Gr}(\sigma_k^f)$ is a sequence of η-asymptotically holomorphic embeddings in $\mathrm{Bi}(r_e, r_f, N)$.*
3. *Denoting by V_{ϵ^-} the compact submanifold of V obtained by removing an ϵ-neighborhood of W, we obtain that ϕ_k is η-transverse to V_{ϵ^-}.*

Moreover the result can be extended to continuous one-parameter families of sections $(s_{k,t}^e)_{t\in[0,1]}$ and $(s_{k,t}^f)_{t\in[0,1]}$ providing embeddings to the bigrassmannian and to continuous one-parameter families of open submanifolds V_t. Thus we obtain continuous families of sequences $\sigma_{k,t}^e$ and $\sigma_{k,t}^f$ verifying the required conditions. □

5.2. **Dependence loci of sections of a vector bundle.** Suppose that E is an hermitian vector bundle of rank n and consider $s_1, \dots, s_m$ sections of E. Then we can interpret $s = (s_1, \dots, s_m)$ as a morphism of bundles $s : \underline{\mathbb{C}}^m \to E$. The r-determinantal set of s is

$$\Sigma^r(s) = \{x \in M \big| \dim\,[s_1(x), \dots, s_m(x)] = r\},$$

and it is called the r-dependence locus of the sections $s_1, \dots, s_m$.

Theorem 5.4. *Let (M, ω) be a closed symplectic manifold of integer class and let E be a rank n hermitian vector bundle. Then, for k large enough, there exist $s_k = (s_k^1, \dots, s_k^m)$ sections of $\mathbb{C}^m\otimes E$ such that*

1. *$\Sigma^r(s_k)$ is an open symplectic submanifold of M.*
2. *$\mathrm{codim}\ \Sigma^r(s_k) = 2(m-r)(n-r)$. The set of manifolds $\{\Sigma^r(s_k)\}_r$ constitutes a stratified submanifold.*

Moreover, any two stratified submanifolds constructed by the process in the proof below are isotopic.

Proof. The proof is similar to the arguments developed in Subsection 5.1. Let $\mathcal{U}$ be the universal bundle over $\mathrm{Gr}(n, N)$ and consider m holomorphic sections $s_1, \dots, s_m$ verifying that:

1. $D_r = \Sigma^r(s)$ is an open complex submanifold in $\mathrm{Gr}(n, N)$.
2. $\mathrm{codim}_{\mathbb{C}}\, D_r = (m-r)(n-r)$.

Now we choose a sequence of asymptotically holomorphic embeddings $\phi_k : M \to \mathrm{Gr}(n, N)$ such that $\phi_k^*\mathcal{U} = E\otimes L^{\otimes k}$. If we assure that, for each r, ϕ_k is transverse to D_r with an angle $\epsilon > 0$ independent of k, we have finished the proof because of Proposition 3.10. But we may perturb ϕ_k by using analogues of Lemma 5.2 and Proposition 5.3 for the case of just one grassmannian. □

5.3. **Homology and homotopy groups of the determinantal submanifolds.** In this Subsection we prove a result concerning the topology of smooth determinantal submanifolds analogous to Proposition 39 in [Do96] (symplectic Lefschetz hyperplane theorem) and Proposition 2 in [Au97]. The main result is

Proposition 5.5. *Let E, F be vector bundles of ranks r_e, r_f, respectively, over a closed symplectic manifold (M, ω) of integer class and let D_r^k be a sequence of determinantal submanifolds constructed, by using the vector bundles $E \otimes (L^*)^{\otimes k}$ and $F \otimes L^{\otimes k}$, as a transverse intersection of an asymptotically holomorphic sequence of embeddings in $\mathrm{Bi}(r_e, r_f, N)$ with the determinantal varieties of a fixed generic section s of the universal bundle $\mathcal{U}_{ef}$ over $\mathrm{Bi}(r_e, r_f, N)$. Assume that the stratified determinantal submanifold has only one stratum D_r^k. Then the inclusion $i : D_r^k \to M$ induces, for k large enough, an isomorphism on homotopy groups π_p for $p < \frac{1}{2}\mathrm{dim}\ D_r^k$ and a surjection on π_p for $p = \frac{1}{2}\mathrm{dim}\ D_r^k$. The same property also holds for homology groups.*

Remark that the asumption of only one stratum implies that $r = \min\{r_e, r_f\} - 1$ and $2(r_e - r + 1)(r_f - r + 1) = 4(|r_e - r_f| + 2) > \mathrm{dim}\ M$. Along the proofs we will suppose that $r_e \geq r_f$, leaving the details of the other case to the reader. We proceed in several steps.

5.3.1. *Determinant vector spaces.* Let V, W be vector spaces of dimensions m and n $(m \geq n)$ respectively. We need some results about the behaviour of the determinant vector space $\bigwedge^r (V^*) \otimes \bigwedge^r W$ associated to the vector space of linear morphisms $V^* \otimes W$. We define the r-fold wedge product $\bigwedge^r$ of a linear application $\varphi \in \mathrm{Hom}(V, W)$ as the linear application

$$\begin{array}{rcl} \bigwedge^r \varphi : \bigwedge^r V & \to & \bigwedge^r W \\ v_1 \wedge \cdots \wedge v_r & \to & \varphi(v_1) \wedge \cdots \wedge \varphi(v_r). \end{array}$$

Thus we obtain a non-linear map $\bigwedge^r : \mathrm{Hom}(V, W) \to \mathrm{Hom}(\bigwedge^r V, \bigwedge^r W)$. The previous definition extends in an obvious way to any pair of vector bundles E and F providing a non-linear map of vector bundles $\bigwedge^r : \mathrm{Hom}(E, F) \to \mathrm{Hom}(\bigwedge^r E, \bigwedge^r F)$. With this notation a rank $r - 1$ determinantal submanifold D_{r-1} associated to a morphism φ between vector bundles E and F is the set

$$D_{r-1} = \{x \in M : \bigwedge^r \varphi(x) = 0\}.$$

Lemma 5.6. *Let V, W be vector spaces of dimensions m and n $(m \geq n)$ respectively, then the set $R(V, W) = \bigwedge^n(\mathrm{Hom}(V, W)) - \{0\}$ is a smooth open complex submanifold of $\mathrm{Hom}(\bigwedge^n V, \bigwedge^n W)$ of dimension $m - n + 1$. Moreover $R(V, W)$ is invariant under multiplication by non-zero complex scalars, and so given any point $d \in R(V, W)$ then, using the standard identification between a vector space and its tangent space at a point, $d \in T_d R(V, W)$.*

Proof. The last statement is obvious. For the first one, fix basis $(e_1, \ldots, e_m)$ in V and $(f_1, \ldots, f_n)$ in W. First notice that $R(V, W)$ is invariant under the actions of the groups $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$. Thus for computing $T_{\wedge^n(\varphi)} R(V, W)$ we can restrict our atention to the point

$$\varphi = \sum_{i=1}^{n} f_i \otimes e_i^*. \tag{15}$$

Notice that this is possible since the condition $\bigwedge^n \varphi \neq 0$ implies that the linear map φ has rank n and therefore suitable changes of basis provide the expression (15).

Now, we only have to compute the images of the tangent basis $\varphi_{ij} = \frac{d}{dt}|_{t=0}(\varphi + tb_{ij})$, where $b_{ij} = f_j \otimes e_i^*$. First assume that $i \leq n$, then we obtain

$$(\bigwedge^n)_* \varphi_{ij} = \begin{cases} \varphi, & i = j, \\ 0, & i \neq j \end{cases}$$

However for the cases $i > n$ we obtain

$$(\bigwedge^n)_* \varphi_{ij} = (-1)^{n-j} f_1 \wedge \cdots \wedge f_n \otimes e_1^* \wedge \cdots e_{j-1}^* \wedge e_{j+1}^* \wedge \cdots \wedge e_n^* \wedge e_i^*.$$

Then the image of this tangent basis has dimension $m - n + 1$. This happens at any point of $R(V, W)$. Now, the image of an application of constant rank is locally a submanifold.

Finally we have to check that the counterimages of $\bigwedge^n$ are connected, i.e. given two morphisms φ_0 and φ_1 such that $\bigwedge^n \varphi_0 = \bigwedge^n \varphi_1$ then there exists a path $\{\varphi_t\}_{t\in[0,1]}$ connecting the two morphisms and satisfying $\bigwedge^n \varphi_t = \bigwedge^n \varphi_0$. For this, note that the kernels of φ_0 and φ_1 coincide. Therefore there exists an endomorphism A in $GL(W)$ such that $A\varphi_0 = \varphi_1$. Such A is forced to be in $SL(W)$. Now fix a path A_t, $t \in [0, 1]$, connecting the identity with A and put $\varphi_t = A_t \varphi_0$. □

This Lemma extends trivially to vector bundles to obtain the following

Lemma 5.7. *Let E, F be vector bundles of ranks m and n ($m \geq n$) respectively, then the fibration $R(E, F)$, given at any point $x \in M$ by $\bigwedge^n(\mathrm{Hom}(E_x, F_x)) - \{0\}$, has smooth fibers which are open complex submanifolds of $\mathrm{Hom}(\bigwedge^n E_x, \bigwedge^n F_x)$ of dimension $m - n + 1$. Moreover $R(E, F)$ is invariant under multiplication by a never null complex-valued function, and so given any point $d \in R(E, F)$ we have, using the standard identification between a vector space and its tangent space at a point, that $d \in T_d R(E, F)$.*

5.3.2. *Generalized asymptotically holomorphic sequences of sections of vector bundles.* Now we recall the process of construction of a sequence of symplectic determinantal submanifolds. Let E, F be vector bundles of ranks r_e and r_f, respectively, and suppose $r_e \geq r_f$. Write

$$r = \min\{r_e, r_f\} = r_f.$$

Fix a generic section s of the universal bundle $\mathcal{U}_{ef}$ over $\mathrm{Bi}(r_e, r_f, N)$. We embed M in $\mathrm{Bi}(r_e, r_f, N)$ constructing an asymptotically holomorphic sequence ϕ_k of embeddings. Using Lemma 5.2 and Proposition 5.3 we assure that the sequence is transverse to the holomorphic determinantal varieties defined by s in $\mathrm{Bi}(r_e, r_f, N)$. We can define a sequence of sections of the bundles $E^* \otimes F \otimes L^{\otimes 2k}$ as

$$s_k = \phi_k^* s.$$

We consider now the connection $\hat{\nabla}_k$ defined on $E^* \otimes F \otimes L^{\otimes 2k}$ as the pull-back of the canonical one defined in $\mathcal{U}_{ef}$. Also we consider in M the sequence of metrics $\hat{g}_k$ defined as the pull-back through ϕ_k of the standard metric on the bigrassmanian $\mathrm{Bi}(r_e, r_f, N)$. Then using properties 1 and 2 of Definition 1.1, we obtain that the sequence s_k is asymptotically holomorphic with respect to the fixed complex structure J in M, computing the derivatives respect to $\hat{\nabla}_k$ and the norms respect to $\hat{g}_k$. Analogously taking the pull-back of the connection associated to $\bigwedge^r \pi_e^*(\mathcal{U}_e) \otimes \bigwedge^r \pi_f^*(\mathcal{U}_f)$, we obtain connections for the bundles $\bigwedge^r(E^* \otimes L^{\otimes k}) \otimes \bigwedge^r(F \otimes L^{\otimes k})$. Then the sequence $\bigwedge^r s_k$ is asymptotically J-holomorphic with respect to these connections and to the metric $\hat{g}_k$.

Now we look for a condition to express when the sections $\bigwedge^r s_k$ are transversal in a certain sense. The key property is

Lemma 5.8. *Let E and F be vector bundles with connections ∇^e and ∇^f respectively. Suppose s is a section of the bundle of morphisms $E^* \otimes F$ equipped with*

the connection ∇^{ef} induced by ∇^e and ∇^f. If $\bigwedge^r s(x) \neq 0$ at a point $x \in M$, then $\nabla^{ef} \bigwedge^r s(x) \in T_{\wedge^r s(x)} R(E_x, F_x)$.

Proof. To check this we only have to show that the following diagram is commutative

$$\begin{array}{ccc} \Omega^0(E^* \otimes F) & \stackrel{id \oplus \nabla^{ef}}{\rightarrow} & \Omega^0(E^* \otimes F) \bigoplus \Omega^1(E^* \otimes F) \\ \downarrow \bigwedge^r & & \downarrow T \bigwedge^r \\ \Omega^0(\bigwedge^r (E^*) \otimes \bigwedge^r F) & \stackrel{id \oplus \nabla^{ef}}{\rightarrow} & \Omega^0(\bigwedge^r (E^*) \otimes \bigwedge^r F) \bigoplus \Omega^1(\bigwedge^r (E^*) \otimes \bigwedge^r F). \end{array}$$

The map $T \bigwedge^r$ is defined as

$$\begin{aligned} T \bigwedge^r : \Omega^0(E^* \otimes F) \oplus \Omega^1(E^* \otimes F) &\rightarrow \Omega^0(\bigwedge^r (E^*) \otimes \bigwedge^r F) \oplus \Omega^1(\bigwedge^r (E^*) \otimes \bigwedge^r F) \\ (s_0, s_1) &\mapsto \left(s_0, \lim_{t \to 0} \frac{\bigwedge^r (s_0 + t s_1)}{t} \right). \end{aligned}$$

To check this one fixes local frames in E and F and carries out the computation explicitly. $\square$

Given a generic section s of the bundle of morphisms $E^* \otimes F$ then we denote by D^ϵ_{r-2} the ϵ-neighborhood of the determinantal set D_{r-2} associated to s.

Definition 5.9. *Let E and F be vector bundles over M of ranks r_e and r_f ($r_e \geq r_f$) respectively. Put $r = r_f$. We say that the section s is η-$\wedge^r$-transverse to 0, for some $\eta > 0$, if for any $x \in M - D^\eta_{r-2}$ such that $|\bigwedge^r s(x)| < \eta$ then the covariant derivative $\hat{s}(x) = \nabla \bigwedge^r s(x)$ has rank $r_e - r_f + 1$ and also there exists a right inverse $\theta : T_{\wedge^r s(x)} R^r(E, F) \to T_x M$ of $\hat{s}(x)$ with norm less that η^{-1}.*

We cannot impose the estimated transversality near the stratum D_{r-2} because the section $\bigwedge^r s$ is always critical in that stratum, so if we want to obtain a notion of estimated transversality we need to remove a neighborhood of D_{r-2}.

Observe that given any small $\eta > 0$, the section s is η-$\wedge^r$-transverse to 0.

Using that $\phi_k(M)$ is transverse to D_{r-1} we can check that s_k is η'-$\wedge^r$-transverse to 0 on M, for some universal $\eta' > 0$, with the connections and metrics defined in the prededent lines. Observe that to guarantee this property is absolutely necessary that the minimum distance from $\phi_k(M)$ to D_{r-2} be greater than η, but this is true by construction.

5.3.3. *Proof of Proposition 5.5.* We have as starting data a sequence of asymptotically holomorphic sections of the bundles $E^* \otimes F \otimes L^{\otimes 2k}$ obtained by pull-back of a fixed section s of the universal bundle $\mathcal{U}_{ef}$. As before, we may suppose that $r_e \geq r_f$ and write $r = r_f$. Therefore the only non-empty stratum is D^k_{r-1}, by assumption. We assume also that s_k is η-$\wedge^r$-transverse to 0, for a universal $\eta > 0$. The stratum D_{r-2} is empty and so the η-$\wedge^r$-transversality is checked all over M. We can follow the ideas of [Do96, Au97] to develop the proof.

We define the function $f_k = \log |\bigwedge^r s_k|^2$. Clearly $f_k(-\infty) = D^k_{r-1}$. Denote the complex dimension of D^k_{r-1} by N. We are going to show that all the critical points of f_k are of index at least $N+1$. Therefore a standard Morse-theoretic argument will finish the proof.

Denote $\sigma_k = \bigwedge^r s_k$. First notice that if x is a critical point of $|\sigma_k|^2$ then $\sigma_k(x)$ is not in the image of $\nabla \sigma_k$ and so $\nabla \sigma_k$ is not surjective to $T_{\sigma_k(x)} R(E_x, F_x)$. It follows from the η-$\wedge^r$-transversality property that $|\sigma_k(x)| > \eta$.

Now we differentiate f_k to obtain

$$\partial f_k = \frac{1}{|\sigma_k|^2} (\langle \partial \sigma_k, \sigma_k \rangle + \langle \sigma_k, \bar{\partial} \sigma_k \rangle).$$

At a critical point x, $\partial f_k(x) = 0$. Using the asymptotic holomorphic bounds we obtain

$$|\langle \partial\sigma_k, \sigma_k\rangle| = |\langle \bar\partial\sigma_k, \sigma_k\rangle| \le Ck^{-1/2}|\sigma_k|. \tag{16}$$

Differentiating a second time we obtain, evaluating at a critical point, the expression

$$\bar\partial\partial \log|\sigma|^2 = \frac{1}{|\sigma|^2}(\langle \bar\partial\partial\sigma, \sigma\rangle - \langle \partial\sigma, \partial\sigma\rangle + \langle \bar\partial\sigma, \bar\partial\sigma\rangle + \langle \sigma, \partial\bar\partial\sigma\rangle),$$

where we omit the subindex k for simplicity. Recall that $\bar\partial\partial + \partial\bar\partial$ equals the $(1,1)$-part of the curvature of the bundle $\bigwedge^r(E^*\otimes L^{\otimes k})\otimes\bigwedge^r(F\otimes L^{\otimes k})$. Its $(1,1)$-curvature R is the pull-back through ϕ_k of the $(1,1)$-curvature $\tilde R$ of $\bigwedge^r\mathcal{U}_e\otimes\bigwedge^r\mathcal{U}_f$. So we obtain

$$\bar\partial\partial f_k = \frac{1}{|\sigma|^2}(\langle R\sigma, \sigma\rangle - \langle \partial\bar\partial\sigma, \sigma\rangle + \langle \sigma, \partial\bar\partial\sigma\rangle - \langle \partial\sigma, \partial\sigma\rangle + \langle \bar\partial\sigma, \bar\partial\sigma\rangle).$$

We define the subspace

$$\mathscr{V} = \{v \in T_xM \big| \nabla_v\sigma(x) = \lambda\sigma(x), \text{ for some } \lambda\in\mathbb{C}\}.$$

Using the inequality (16) we obtain, for any $v\in\mathscr{V}$, that

$$|\langle \partial_v\sigma, \sigma\rangle| = |\partial_v\sigma||\sigma| \le Ck^{-1/2}|\sigma|.$$

Restricting $\bar\partial\partial f_k$ to $\mathscr{V}$, it equals to $\frac{1}{|\sigma|^2}\langle R\sigma,\sigma\rangle + O(k^{-1/2})$. Denote the Hessian of f by H_f. We know that $H_f(u)+H_f(Ju) = -2i\bar\partial\partial f_k(u, Ju) = -2i\frac{1}{|\sigma|^2}\langle R(u,Ju)\sigma,\sigma\rangle + O(k^{-1/2})$, for any unit vector $u\in\mathscr{V}$. We claim that it is possible to bound above the expression

$$-2i\frac{1}{|\sigma|^2}\langle R(u,Ju)\sigma,\sigma\rangle \tag{17}$$

by a universal strictly negative constant, where u is a unitary vector. For this we need to estimate the curvature R. We start by computing the curvature of the universal bundle $\mathcal{U}$ over the grassmannian $\mathrm{Gr}(r,N)$. We use the local expression of the curvature of $\mathcal{U}^*$ from [We73, page 82],

$$R_{\mathcal{U}^*} = h^{-1}\overline{df}^t\wedge df - h^{-1}\overline{df}^t f h^{-1}\wedge \bar f^t df,$$

where $f = (f_1,\dots,f_r)$ is a frame in an open neighborhood of $\mathrm{Gr}(r,N)$ and $h = \bar f^t f$. We may assume that we are at the point $\Pi_0 = [\boldsymbol{I}|\boldsymbol{0}]$ of the grassmannian, after suitable change of coordinates. Select the following holomorphic local frame,

$$f = ((1,0,\overset{(r-1)}{\dots},0,z_{11},\dots,z_{1,n-r}),\dots,(0,\dots,0,1,z_{r1},\dots,z_{r,n-r})),$$

So at the point Π_0 we obtain $R_{\mathcal{U}^*} = \overline{df}^t\wedge df$ and

$$R_{\mathcal{U}} = df^t\wedge\overline{df}.$$

In the trivialization (z_{jk}) we take the standard basis $e_{jk} = \frac{\partial}{\partial z_{jk}}$. We obtain $R_{\mathcal{U}}(e_{jk}, ie_{jk}) = -ib_{jj}$, where the endomorphism b_{jj} is defined as $e_j\otimes e_j^*$. So the endomorphism $-iR_{\mathcal{U}}(u,Ju)$ is semi-definite negative for $u\in T_{\Pi_0}\mathrm{Gr}(r,N)$ non-zero. This implies also that $-iR_{\bigwedge^k\mathcal{U}}(u,Ju)$ is semi-definite negative, for $1\le k\le r$. Moreover computing $-iR_{\bigwedge^r\mathcal{U}}(u,Ju)$ in $\mathrm{Gr}(r,N)$, or recalling that $\mathcal{U}$ is very ample, we obtain that this endomorphism is definite negative. Returning to $\mathrm{Bi}(r_e,r_f,N)$ with $r = r_f\le r_e$, we have that the curvature of $\bigwedge^r\mathcal{U}_e\otimes\bigwedge^r\mathcal{U}_f$ is

$$\tilde R = R_{\pi_e^*\bigwedge^r\mathcal{U}_e}\otimes\boldsymbol{I}_1 + \boldsymbol{I}_\nu\otimes R_{\pi_f^*\bigwedge^r\mathcal{U}_f},$$

where $\nu = \begin{pmatrix} r_e \\ r\end{pmatrix}$. So $\tilde R(u,Ju)$ is definite negative, for $u\in T\mathrm{Bi}(r_e,r_f,N)$ unitary vector. Using that the sequence of embeddings $\phi_k = (\phi_k^e,\phi_k^f)$ satisfies properties 1 and 2 of Definition 1.1, we get that the expression (17) is bounded above by a universal strictly negative number.

Therefore, for any unitary $u \in \mathscr{V}$, $H_f(u) + H_f(Ju)$ is negative for k large enough. Recall that from the definition we obtain that $\dim \mathscr{V} \geq 2N+2$. Suppose that there exists a subspace $P \in T_xM$ of real dimension at least $2n - N$ such that H_f in non-negative. The dimension of $P \bigcap JP$ is at least $2n - 2N$, and there the function $H_f(\cdot) + H_f(J\cdot)$ is, obviously, non-negative. Therefore $P \bigcap JP$ has to intersect trivially with $\mathscr{V}$ but $\dim P \bigcap JP + \dim \mathscr{V} \geq 2n+2$, and this is clearly impossible. So such space P does not exist and then the index of f_k at x is greater than N. This finishes the proof. $\square$

5.4. **Chern classes of the constructed submanifolds.** For computing the Chern classes of determinantal submanifolds, we shall use the results of Harris and Tu in [HT84]. All their results are stated for holomorphic determinantal submanifolds in a holomorphic manifold, but they apply without the condition of integrability of the complex structure. We state the formulas that we shall use. Following Subsection 5.1 we denote $r_e = \operatorname{rank} E$, $r_f = \operatorname{rank} F$, $2n = \dim M$ and D_r is the r-determinantal loci of a bundle map $\varphi : E \to F$ constructed in Theorem 1.6. First of all, set

$$\Delta_{i_1,\dots,i_{r_e-r}} = \begin{vmatrix} c_{r_f-r+i_1} & c_{r_f-r+i_1+1} & \cdots & \\ c_{r_f-r+i_2-1} & c_{r_f-r+i_2} & \cdots & \\ & & \ddots & \\ & & \cdots & c_{r_f-r+i_{r_e-r}} \end{vmatrix}, \tag{18}$$

where $c_j = c_j(F-E)$. For instance, $\Delta_{0,\dots,0} = \Delta = \mathrm{PD}([D_r])$, which is the classical Porteous formula for the homology class of a determinantal locus. We can suppose that the indices i_j are decreasing, and so if we have any index $i_j = 0$ we do not write it, e.g. $\Delta_{2,1,0} = \Delta_{2,1}$.

In [HT84] a complete description of the Chern numbers of the tangent bundle of a determinantal submanifold is performed, supposing that $D_{r-1} = \emptyset$ and so D_r is smooth. We concentrate ourselves in the cases $\dim_{\mathbb{C}} D_r = 1$ and $\dim_{\mathbb{C}} D_r = 2$, where Harris and Tu obtain the following formulas:

1. For $\dim_{\mathbb{C}} M = (r_e - r)(r_f - r) + 1$, then $\dim_{\mathbb{C}} D_r = 1$. We have
$$n_1(D_r) = \langle c_1(D_r), [D_r]\rangle = (c_1(M) + (r_e - r)c_1(E-F))\Delta + (r_e - r_f)\Delta_1.$$
2. For $\dim_{\mathbb{C}} M = (r_e - r)(r_f - r) + 2$, then $\dim_{\mathbb{C}} D_r = 2$. We have

$$\begin{aligned}
n_{11}(D_r) &= \langle c_1^2(D_r), [D_r]\rangle = (c_1(M) + (r_e - r)c_1(E-F))^2 \cdot \Delta + \\
&+ 2(r_e - r_f)(c_1(M) + (r_e - r)c_1(E-F)) \cdot \Delta_1 + (r_e - r_f)^2(\Delta_2 + \Delta_{11}), \\
n_2(D_r) &= \langle c_2(D_r), [D_r]\rangle = \big(c_2(M) + (r_e - r)c_1(M)c_1(E-F) + \\
&+ (r_e - r)(c_2(E) - c_2(F)) + \binom{r_e - r}{2} c_1^2(E) - (r_e - r)^2 c_1(E)c_1(F) + \\
&+ \binom{r_e - r + 1}{2} c_1^2(F)\big)\Delta + \\
&+ ((r_e - r)c_1(M) + ((r_e - r)(r_e - r_f) - 1)c_1(E-F))\Delta_1 \\
&+ \frac{1}{2}((r_e - r_f)^2 + (r_e - r) + (r_f - r) - 2)\Delta_2 + \\
&+ \frac{1}{2}((r_e - r_f)^2 - (r_e - r) - (r_f - r) - 2)\Delta_{11}.
\end{aligned}$$

In our case, we are going to apply the above formulas to morphisms $\varphi : E \otimes (L^*)^{\otimes k} \to F \otimes L^{\otimes k}$. We have the following asymptotic expansions for Chern classes

(we write $\omega_k = \frac{k\omega}{2\pi}$ for simplicity)

$$
\begin{aligned}
c_p(F \otimes L^{\otimes k}) &= \binom{r_f}{p} \omega_k^p + O(k^{p-1}), \\
c_p(E \otimes (L^*)^{\otimes k}) &= \binom{r_e}{p} (-\omega_k)^p + O(k^{p-1}), \\
c_p = c_p(F \otimes L^{\otimes k} - E \otimes (L^*)^{\otimes k}) &= \mathrm{Coeff}_{x^p} \frac{(1+x)^{r_f}}{(1-x)^{r_e}} \omega_k^p + O(k^{p-1}) = \\
&= \sum_{i=0}^{r_f} \binom{r_f}{i} \binom{r_e + p - i - 1}{p - i} \omega_k^p + O(k^{p-1}). \qquad (19)
\end{aligned}
$$

We are going to give two families of examples to show that the symplectic manifolds obtained here are more general than those in [Au97].

5.4.1. *Example 1.* Choose $\dim_{\mathbb{C}} M = (r_e - r)(r_f - r) + 1$ and so we can apply the formulas for the complex 1-dimensional case. Also suppose that $r = 1$ and $r_e = 2$, so $\dim_{\mathbb{C}} M = r_f = n > 1$. By Proposition 5.5 the submanifolds D_1 are connected. Now $\mathrm{PD}[D_1] = \Delta = c_{n-1}$ and $\Delta_1 = c_n$. Using (19) we get that

$$
\begin{aligned}
\mathrm{vol}_{\omega_k}(D_1) &= \Delta\omega_k = (n2^{n-1} + O(k^{-1}))\mathrm{vol}_{\omega_k}(M), \\
n_1(D_1) &= -(n+2)\omega_k\Delta + (2-n)\Delta_1 + O(k^{-1})\mathrm{vol}_{\omega_k}(M) = \\
&= (-(n+2)n2^{n-1} + (2-n)(n2^{n-1} + 2^n) + O(k^{-1}))\mathrm{vol}_{\omega_k}(M) \\
\frac{n_1(D_1)}{\mathrm{vol}_{\omega_k}(D_1)} &= -2 - 2n + \frac{4}{n} + O(k^{-1}).
\end{aligned}
$$

To compare with the Auroux' case we compute the precedent symplectic invariants for this situation. Denote by Z the zero set of a transverse section of a bundle of the form $E \otimes L^{\otimes k}$, we choose $\mathrm{rank}\, E = n - 1$ to set up the comparison. Suppose that Z is symplectic. Using Proposition 5 in [Au97] we obtain

$$
\begin{aligned}
\mathrm{vol}_{\omega_k}(Z) &= (1 + O(k^{-1}))\mathrm{vol}_{\omega_k}(M), \\
n_1(Z) &= (1 - n + O(k^{-1}))\mathrm{vol}_{\omega_k}(M), \\
\frac{n_1(Z)}{\mathrm{vol}_{\omega_k}(Z)} &= 1 - n + O(k^{-1}).
\end{aligned}
$$

Therefore there does not exist any $n \geq 2$ such that the quotients $\frac{n_1(D_1)}{\mathrm{vol}_{\omega_k}(D_1)}$ coincide with the quotients $\frac{n_1(Z)}{\mathrm{vol}_{\omega_k}(Z)}$, obviously for k large enough. So Auroux' sequences of submanifold are not symplectomorphic to our sequences of determinantal submanifolds.

To check that, for k large, our determinantal submanifolds do not coincide with Auroux' examples we work as follows. Suppose that for integers k_1, k_2 the submanifold $D_1 = D_1^{k_1}$ is isotopic to $Z = Z_{k_2}$. Then they define the same cohomology class and hence $n2^{n-1}k_1 = k_2 + O(1)$. Also $n_1(D_1) = n_1(Z)$ implies $(-2 - 2n + \frac{4}{n})k_1 = (1-n)k_2 + O(1)$. So, for large enough k's, $(1-n)n2^{n-1} = -2 - 2n + \frac{4}{n}$ and hence $n = 2$. Therefore for $n > 2$ and large k we get new examples of symplectic submanifolds.

Note that for $n = r_e = r_f = 2$, the determinantal set D_1 for a morphism $\varphi : E \otimes (L^*)^{\otimes k} \to F \otimes L^{\otimes k}$ is the zero set of the section $\bigwedge^2 \varphi$ of $\bigwedge^2 E^* \otimes \bigwedge^2 F \otimes L^{\otimes 4k}$. Since this zero set is smooth of the expected codimension, our example is just one of Auroux' examples.

5.4.2. *Example 2.* Now, choose $\dim_{\mathbb{C}} M = (r_e - r)(r_f - r) + 2$ and so we can apply the formulas for the complex 2-dimensional case. Again we suppose that $r = 1$ and $r_e = 2$, so $\dim_{\mathbb{C}} M = r_f + 1 = n > 2$. By Proposition 5.5 these submanifolds are connected. In this case we have

$$\begin{aligned}
\mathrm{vol}_{\omega_k}(D_1) &= ((n-1)2^{n-2} + O(k^{-1}))\mathrm{vol}_{\omega_k}(M), \\
n_{11}(D_1) &= (4(n-1)(n^2-5)2^{n-2} + O(k^{-1}))\mathrm{vol}_{\omega_k}(M) \\
n_2(D_1) &= (2(n^2+n-4)(n-1)2^{n-2} + O(k^{-1})\mathrm{vol}_{\omega_k}(M) \\
\frac{n_2(D_1)}{n_{11}(D_1)} &= \frac{n^2+n-4}{2(n^2-5)} + O(k^{-1}).
\end{aligned}$$

For the Auroux' case with $\mathrm{rank}\, E = n - 2$ we obtain

$$\begin{aligned}
\mathrm{vol}_{\omega_k}(Z) &= (1 + O(k^{-1}))\mathrm{vol}_{\omega_k}(M), \\
n_{11}(Z) &= ((n-2)^2 + O(k^{-1}))\mathrm{vol}_{\omega_k}(M), \\
n_2(Z) &= \left(\frac{(n-1)(n-2)}{2} + O(k^{-1})\right)\mathrm{vol}_{\omega_k}(M) \\
\frac{n_2(Z)}{n_{11}(Z)} &= \frac{n-1}{2(n-2)} + O(k^{-1}).
\end{aligned}$$

If we compute the symplectic invariants $\frac{n_{11}(Z)}{\mathrm{vol}_{\omega_k}(Z)}$ and $\frac{n_2(Z)}{\mathrm{vol}_{\omega_k}(Z)}$, it is easy to verify that Auroux' submanifolds are not symplectomorphic to the determinantal ones constructed in this example.

Moreover, for 4-manifolds, the numbers $n_2 = \chi$ and $n_{11} = (2\chi + 3\sigma)/4$ are topological invariants. Therefore $\frac{n_2}{n_{11}}$ is a topological invariant. Comparing the Auroux' case and the determinantal example we find that these symplectic submanifolds are not even *homeomorphic*, for k large enough (even choosing different k's in either case).

In general, it is clear that the determinantal class is quite bigger than the Donaldson-Auroux one. We could compute more examples and more precise invariants using recent results from algebraic geometry about the topology of determinantal submanifolds. As a reference it could be useful [HT84b, Pr88, PP91]. Remark that in these references the computations are performed even in the singular case. To adapt them to the symplectic category we would need to define the Segre classes of a singular symplectic manifold. This definition seems quite natural.

References

[Ar82] V. I. Arnold, S. M. Gusein-Zadeh, A. N. Varchenko, *Singularities of smooth mappings.* Monographs in Mathematics. Birkhäuser. Boston. (1982).

[Au97] D. Auroux. *Asymptotically holomorphic families of symplectic submanifolds.* Geom. Funct. Anal., **7**, 971-995 (1997).

[Au99] D. Auroux. *Symplectic* 4*-manifolds as branched coverings of* $\mathbb{CP}^2$. To appear in Invent. Math.

[BU99] D. Bortwick, A. Uribe. *Nearly kählerial embeddings of symplectic manifolds.* To appear in Invent. Math.

[Do96] S. K. Donaldson. *Symplectic submanifolds and almost-complex geometry.* J. Diff. Geom., **44**, 666-705 (1996).

[Do98] S. K. Donaldson. *Lefschetz fibrations in Symplectic Geometry.* Doc. Math. Extra Vol. ICM 98, II, 309-314 (1998).

[Do99] S. K. Donaldson. *Lefschetz pencils in Symplectic Geometry.* Preprint (1999).

[GH78] P. Griffiths, J. Harris, *Principles of Algebraic Geometry.* Wiley-Interscience. New York. (1978)

[Gr86] M. Gromov. *Partial Differential Relations.* Ergebnisse Math. (3) **9**. Springer (1986).

[HT84] J. Harris, L. Tu. *Chern numbers of kernel and cokernel bundles.* Invent. Math., **75**, 467-475 (1984).

[HT84b] J. Harris, L. Tu. *On symmetric and skew-symmetric determinantal varieties.* Topology, **23**, 71-84. (1984).

[IMP99] A. Ibort, D. Martínez, F. Presas. *On the construction of contact submanifolds with prescribed topology.* Preprint (1999). Universidad Complutense de Madrid.

[MS94] D. McDuff, D. Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs (1994).

[Pa99] R. Paoletti. *Symplectic submanifolds over projective fibrations of symplectic manifolds.* To appear in Annales de l'Institut Fourier. Grenoble.

[Pa99b] R. Paoletti. *Symplectic submanifolds in special position.* Preprint (1999).

[PP91] A. Parusinski, P. Pragasz. *Characteristic numbers of degeneracy loci.* Contemporary Mathematics, **123** (1991).

[Pr88] P. Pragasz. *Enumerative geometry of degeneracy loci.* Ann. Sci. École Norm Sup., **21**, 413-454 (1988).

[Ti90] G. Tian. *On a set of polarized metrics on algebraic manifolds.* J. Diff. Geom., **32**, 99-130 (1990).

[Ti77] D. Tischler. *Closed 2-forms and an embedding theorem.* J. Diff. Geom., **12**, 229-235 (1977).

[Vo78] J. A. Vogelaar, *Constructing vector bundles from codimension-two subvarieties.* Proefscrift. Rijksuniversitteit te Leiden. (1978).

[We73] R. Wells, *Differential Analysis on Complex Manifolds,* Prentice-Hall, Englewood Cliffs, N.J., 2nd ed. Springer-Verlag. (1973).

[Yo83] B. Yomdin. *The geometry of critical and near- critical values of differentiable mappings.* Math. Annalen, **104**, 495-515 (1983).